

Geometric realization of the local Langlands correspondence for representations of conductor three

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Abstract

We prove a realization of the local Langlands correspondence for two-dimensional representations of a Weil group of conductor three in the cohomology of Lubin-Tate curves by a purely local geometric method.

Introduction

Let K be a non-archimedean local field with a finite residue field k of characteristic p . Let \mathfrak{p} be the maximal ideal of the ring of integers of K . We take an algebraic closure K^{ac} of K . Let K^{ur} be the maximal unramified extension of K inside K^{ac} . Let \widehat{K}^{ac} and \widehat{K}^{ur} denote the completion of K^{ac} and K^{ur} respectively. For a natural number n , we write $\text{LT}(\mathfrak{p}^n)$ the Lubin-Tate curve with full level n structure over \widehat{K}^{ur} . We write W_K for the Weil group of K . Let D be the central division algebra over K of invariant $1/2$. Let ℓ be a prime number different from p . We take an algebraic closure $\overline{\mathbb{Q}}_\ell$ of \mathbb{Q}_ℓ . Then the groups W_K , $GL_2(K)$ and D^\times act on $\varinjlim_n H_c^1(\text{LT}(\mathfrak{p}^n)_{\widehat{K}^{\text{ac}}}, \overline{\mathbb{Q}}_\ell)$, and these actions partially realize the local Langlands correspondence and the local Jacquet-Langlands correspondence for GL_2 . This realization was proved by Carayol in [Ca] using global automorphic arguments. However there is no known proof using only a local geometric method.

In this paper, we give a purely local proof of a realization of the local Langlands correspondence for two-dimensional W_K -representations of conductor three, using a calculation of a semi-stable reduction of a Lubin-Tate curve in [IT]. We note that three is the smallest integer which is a conductor of a primitive two-dimensional W_K -representation. The calculation in [IT] is done by purely local methods. One more ingredient of the proof is a result by Mieda in [M2], where a realization of the local Jacquet-Langlands correspondence is proved by purely local methods. In [IT], the actions of W_K and D^\times on a Lubin-Tate curve are calculated in some sense. Using the calculation, we can study representations of W_K and D^\times in the cohomology of Lubin-Tate curves. On the other hand, we have already known relation between representations of $GL_2(K)$ and D^\times in the cohomology by the result of [M2]. Therefore, we can study the relation between representations of W_K and $GL_2(K)$ in the cohomology. This enables us to show a realization of the local Langlands correspondence in the cohomology of Lubin-Tate curves.

In the study of a realization of the local Langlands correspondence for GL_2 , the most difficult and interesting case is the dyadic case, which means the case where $p = 2$. A proof in the case where p is odd is similar and easier. Therefore, we have decided to write a proof only in the dyadic case.

In the dyadic case, the irreducible two-dimensional W_K -representations of conductor three are primitive. Construction of the local Langlands correspondence for primitive representations in [Ku] uses Weil representations (cf. [BH, §12]). On the other hand, our descriptions of representations of $GL_2(K)$ in the cohomology of Lubin-Tate curves are given by cuspidal types. Therefore, it is a non-trivial problem to check that the described representations in the cohomology actually give the local Langlands correspondence.

We explain the contents of this paper. In Section 1, we recall definitions of Lubin-Tate curves, and give an easy consequence of a result in [M2]. In Section 2, we recall a calculation of a semi-stable reduction of a Lubin-Tate curve in [IT].

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In Section 3, we study cohomology of an elliptic curve, which appears in the semi-stable reduction of the Lubin-Tate curve. The cohomology of this elliptic curve gives a primitive W_K -representation of conductor three. We give also an explicit description of this primitive representation.

In Section 4, we show that a correspondence of explicitly described representations appear in the cohomology of Lubin-Tate curves. In Section 5, we show that the correspondence obtained in Section 4 is actually the local Langlands correspondence by calculating epsilon factors. In other word, we give a description of the local Langlands correspondence via cuspidal type for representation of conductor three. To determine the sign of an epsilon factor, we calculate Artin map for a wildly ramified abelian extension with a non-trivial ramification filtration.

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Notation

In this paper, we use the following notation. For a field F , an algebraic closure of F is denoted by F^{ac} and a separable closure of F is denoted by F^{sep} . Let K be a non-archimedean local field. Let \mathcal{O}_K denote the ring of integers and k the residue field of characteristic $p > 0$. Let \mathfrak{p} be the maximal ideal of \mathcal{O}_K . We fix a uniformizer ϖ of K . Let $q = |k|$. For any finite extension F of K , let G_F denote the absolute Galois group of F , W_F denote the Weil group of F and I_F denote the inertia subgroup of W_F . Let K^{ur} denote the maximal unramified extension of K in K^{ac} . The completion of K^{ac} and K^{ur} is denoted by \widehat{K}^{ac} and \widehat{K}^{ur} respectively. We write $\mathcal{O}_{\widehat{K}^{\text{ac}}}$ and $\mathcal{O}_{\widehat{K}^{\text{ur}}}$ for the ring of integers of \widehat{K}^{ac} and \widehat{K}^{ur} respectively. For an element $a \in \mathcal{O}_{\widehat{K}^{\text{ac}}}$, we write \bar{a} for the image of a by the reduction map $\mathcal{O}_{\widehat{K}^{\text{ac}}} \rightarrow k^{\text{ac}}$. Let $v(\cdot)$ denote a valuation of K^{ac} such that $v(\varpi) = 1$. Let $|\cdot|_K$ be the absolute value of K such that $|\varpi|_K = q^{-1}$. For $a, b \in K^{\text{ac}}$ and a rational number $\alpha \in \mathbb{Q}_{\geq 0}$, we write $a \equiv b \pmod{\alpha}$ if we have $v(a - b) \geq \alpha$. For a local ring A , the maximal ideal of A is denoted by m_A . For an irreducible reduced curve X over k^{ac} , we denote by X^c the smooth compactification of X , and the genus of X means the genus of X^c . For an affinoid \mathbf{X} , we write $\overline{\mathbf{X}}$ for its reduction. The category of sets is denoted by **Set**. For a representation π of a group, the dual representation of π is denoted by π^* . We fix $\varpi_0 \in K^{\text{ac}}$ such that $\varpi_0^{6(q-1)} = \varpi$, and $\varpi^{m/(6(q-1))}$ denotes ϖ_0^m for any integer m .

1 Lubin-Tate curve

Let n be a natural number. We put

$$K_1(\mathfrak{p}^n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathcal{O}_K) \mid c \equiv 0, d \equiv 1 \pmod{\mathfrak{p}^n} \right\}.$$

In the following, we define the connected Lubin-Tate curve $\mathbf{X}_1(\mathfrak{p}^n)$ with level $K_1(\mathfrak{p}^n)$.

Let Σ denote the unique (up to isomorphism) formal \mathcal{O}_K -module of dimension 1 and height 2 over k^{ac} . Let \mathcal{C} be the category of Noetherian complete local $\mathcal{O}_{\widehat{K}^{\text{ur}}}$ -algebras with residue field k^{ac} . For a one-dimensional formal \mathcal{O}_K -module $\mathcal{F} = \text{Spf } A[[X]]$ over $A \in \mathcal{C}$ and an element $a \in \mathcal{O}_K$, we write $[a]_{\mathcal{F}}(X) \in A[[X]]$ for the a -multiplication on \mathcal{F} . For a formal one-dimensional \mathcal{O}_K -module $\mathcal{F} = \text{Spf } A[[X]]$ over $A \in \mathcal{C}$ and an A -valued point P of \mathcal{F} , the corresponding element of m_A is denoted by $x(P)$. We consider the functor

$$\mathcal{A}_1(\mathfrak{p}^n): \mathcal{C} \rightarrow \mathbf{Set}; A \mapsto [(\mathcal{F}, \iota, P)],$$

where \mathcal{F} is a formal \mathcal{O}_K -module over A with an isomorphism $\iota: \Sigma \simeq \mathcal{F} \otimes_A k^{\text{ac}}$ and P is a ϖ^n -torsion point of \mathcal{F} such that

$$\prod_{a \in \mathcal{O}_K/\mathfrak{p}^n} (X - x([a]_{\mathcal{F}}(P))) \mid [\varpi^n]_{\mathcal{F}}(X)$$

in $A[[X]]$. This functor is represented by a regular local ring $\mathcal{R}_1(\mathfrak{p}^n)$. We write $\mathfrak{X}_1(\mathfrak{p}^n)$ for $\mathrm{Spf} \mathcal{R}_1(\mathfrak{p}^n)$. Its generic fiber is denoted by $\mathbf{X}_1(\mathfrak{p}^n)$, which we call the connected Lubin-Tate curve with level $K_1(\mathfrak{p}^n)$. The space $\mathbf{X}_1(\mathfrak{p}^n)$ is a rigid analytic curve over $\widehat{K}^{\mathrm{ur}}$. We can define the connected Lubin-Tate curve $\mathbf{X}(\mathfrak{p}^n)$ with full level n structure by changing P to be an \mathcal{O}_K -module homomorphism $\phi: (\mathcal{O}_K/\mathfrak{p}^n)^2 \rightarrow m_A$ such that

$$\prod_{a \in (\mathcal{O}_K/\mathfrak{p}^n)^2} (X - \phi(a)) \mid [\varpi^n]_{\mathcal{F}}(X)$$

in $A[[X]]$. We can define also the Lubin-Tate curve $\mathrm{LT}(\mathfrak{p}^n)$ with full level n structure by changing further \mathcal{C} to be the category of $\mathcal{O}_{\widehat{K}^{\mathrm{ur}}}$ -algebras where ϖ is nilpotent, and ι to be a quasi-isogeny $\Sigma \otimes_{k^{\mathrm{ac}}} A/\mathfrak{p}A \rightarrow \mathcal{F} \otimes_A A/\mathfrak{p}A$. We consider $\mathbf{X}(\mathfrak{p}^n)$ and $\mathrm{LT}(\mathfrak{p}^n)$ as rigid analytic curves over $\widehat{K}^{\mathrm{ur}}$.

Let D be the central division algebra over K of invariant $1/2$. We write \mathcal{O}_D for the ring of integers of D . For a positive integer m , let K_m be the unramified extension of K of degree m and k_m be the finite extension over k of degree m . Let $\kappa \in \mathrm{Gal}(K_2/K)$ be the non-trivial element. The ring \mathcal{O}_D has the following description; $\mathcal{O}_D = \mathcal{O}_{K_2} \oplus \varphi \mathcal{O}_{K_2}$ with $\varphi^2 = \varpi$ and $a\varphi = \varphi a^{\kappa}$ for $a \in \mathcal{O}_{K_2}$. We define an action of \mathcal{O}_D on Σ by $\zeta(X) = \bar{\zeta}X$ for $\zeta \in \mu_{q^2-1}(\mathcal{O}_{K_2})$ and $\varphi(X) = X^q$. Then this give an isomorphism $\mathcal{O}_D \simeq \mathrm{End}(\Sigma)$ by [GH, Proposition 13.10]. By this isomorphism, we can define a left action of \mathcal{O}_D^\times on $\mathbf{X}_1(\mathfrak{p}^n)$ and $\mathbf{X}(\mathfrak{p}^n)$.

Let ℓ be a prime number different from p . We take an algebraic closure $\overline{\mathbb{Q}}_\ell$ of \mathbb{Q}_ℓ . We put $H_{\mathrm{LT}}^1 = \varinjlim_n H_c^1(\mathrm{LT}(\mathfrak{p}^n)_{\widehat{K}^{\mathrm{ac}}}, \overline{\mathbb{Q}}_\ell)$. Then we can define an action of $GL_2(K) \times W_K \times D^\times$ on H_{LT}^1 (cf. [Da, 3.2, 3.3]).

We write $\mathrm{Irr}(D^\times, \overline{\mathbb{Q}}_\ell)$ for the set of isomorphism classes of irreducible smooth representations of D^\times over $\overline{\mathbb{Q}}_\ell$, and $\mathrm{Disc}(GL_2(K), \overline{\mathbb{Q}}_\ell)$ for the set of isomorphism classes of irreducible discrete series representations of $GL_2(K)$ over $\overline{\mathbb{Q}}_\ell$. Let $\mathrm{LJ}: \mathrm{Irr}(D^\times, \overline{\mathbb{Q}}_\ell) \rightarrow \mathrm{Disc}(GL_2(K), \overline{\mathbb{Q}}_\ell)$ be the local Jacquet-Langlands correspondence. We denote by LJ the inverse of LJ . Let $\mathrm{Nrd}_{D/K}: D^\times \rightarrow K^\times$ be the reduced norm map, and $\mathrm{Trd}_{D/K}: D \rightarrow K$ be the reduced trace map. The following proposition follows from a result in [M2], which is proved by local methods. We note that the characteristic of a local field is assumed to be zero in [M2], but the same proof works in the equal characteristic case.

Proposition 1.1. *For a cuspidal representation π of $GL_2(K)$ over $\overline{\mathbb{Q}}_\ell$, we have*

$$\mathrm{Hom}_{GL_2(K)}(H_{\mathrm{LT}}^1, \pi) \simeq \mathrm{LJ}(\pi)^{\oplus 2}$$

as representations of D^\times .

Proof. Let ω_π be the central character of π . We take $c_\pi \in \overline{\mathbb{Q}}_\ell$ such that $c_\pi^2 = \omega_\pi(\varpi)$. We define a character ζ_π of $GL_2(K)$ by $\zeta_\pi(g) = c_\pi^{v(\det(g))}$, and a character ξ_π of D^\times by $\xi_\pi(d) = c_\pi^{v(\mathrm{Nrd}_{D/K}(d))}$ for $d \in D^\times$. We put

$$H_{\mathrm{LT}, \varpi}^i = \varinjlim_n H_c^i((\mathrm{LT}(\mathfrak{p}^n)/\varpi^{\mathbb{Z}})_{\widehat{K}^{\mathrm{ac}}}, \overline{\mathbb{Q}}_\ell)$$

for non-negative integer i . Then we have

$$\mathrm{Hom}_{GL_2(K)}(H_{\mathrm{LT}}^1, \pi) \simeq \mathrm{Hom}_{GL_2(K)}(H_{\mathrm{LT}, \varpi}^1, \pi \otimes \zeta_\pi^{-1}) \otimes \xi_\pi$$

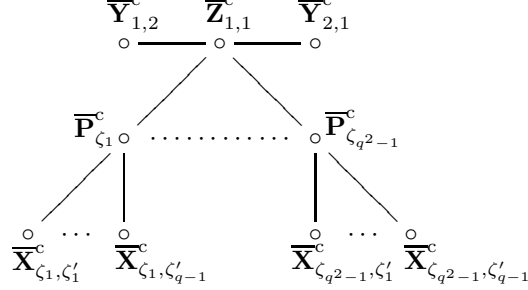
as representations of D^\times by arguments in [St, 3.3]. In the Grothendieck group of finite-dimensional smooth representations of D^\times , we have

$$\sum_{i,j \geq 0} (-1)^{i+j-1} \mathrm{Ext}_{GL_2(K)/\varpi^{\mathbb{Z}}}^j(H_{\mathrm{LT}, \varpi}^i, \pi \otimes \zeta_\pi^{-1}) \otimes \xi_\pi = \mathrm{Hom}_{GL_2(K)}(H_{\mathrm{LT}, \varpi}^1, \pi \otimes \zeta_\pi^{-1}) \otimes \xi_\pi$$

by [M1, Theorem 3.7] and the injectivity of cuspidal representations. Therefore, the claim follows from [M2, Definition 6.2 and Theorem 6.6], because $\mathrm{LJ}(\pi \otimes \zeta_\pi^{-1}) \otimes \xi_\pi = \mathrm{LJ}(\pi)$. \square

2 Semi-stable reduction of $\mathbf{X}_1(\mathfrak{p}^3)$

From now on, we assume that $p = 2$. The dual graph of a semi-stable reduction of $\mathbf{X}_1(\mathfrak{p}^3)$ in the case where q is even is the following:



where $k_2^\times = \{\zeta_1, \dots, \zeta_{q^2-1}\}$, $k^\times = \{\zeta'_1, \dots, \zeta'_{q-1}\}$, $\overline{\mathbf{Y}}_{1,2}$ and $\overline{\mathbf{Y}}_{2,1}$ are defined by $x^q y - xy^q = 1$, $\overline{\mathbf{Z}}_{1,1}^c$ and $\overline{\mathbf{P}}_\zeta^c$ are isomorphic to $\mathbb{P}_{k^{\text{ac}}}^1$, and $\overline{\mathbf{X}}_{\zeta, \zeta'}^c$ are defined by $z^2 + z = w^3$.

For a finite extension K' of K , let $\text{Art}_{K'}: K'^\times \xrightarrow{\sim} W_{K'}^{\text{ab}}$ be the Artin reciprocity map normalized so that the image by $\text{Art}_{K'}$ of a uniformizer is a lift of the geometric Frobenius. We define a homomorphism $|\cdot|: W_K \rightarrow \mathbb{Q}_{>0}$ by the composition

$$W_K \twoheadrightarrow W_K^{\text{ab}} \xrightarrow{\text{Art}_K^{-1}} K^\times \xrightarrow{|\cdot|_K} \mathbb{Q}_{>0}.$$

We put $\mathcal{S} = k_2^\times \times k^\times$ and

$$(W_K \times D^\times)^0 = \{(\sigma, d) \in W_K \times D^\times \mid |\text{Nrd}_{D/K}(d)|_K \cdot |\sigma| = 1\}.$$

Then $(W_K \times D^\times)^0$ acts on $\mathbf{X}_1(\mathfrak{p}^3)$. It induces an action of $(W_K \times D^\times)^0$ on $\coprod_{(\zeta, \zeta') \in \mathcal{S}} \overline{\mathbf{X}}_{\zeta, \zeta'}^c$. For $\sigma \in W_K$, let r_σ be an integer such that $|\sigma| = q^{-r_\sigma}$. Let $\mathcal{O}_D^\times \rtimes W_K$ be a semidirect product where $\sigma \in W_K$ acts on \mathcal{O}_D^\times by $d \mapsto \varphi^{r_\sigma} d \varphi^{-r_\sigma}$. Then we have an isomorphism

$$\mathcal{O}_D^\times \rtimes W_K \simeq (W_K \times D^\times)^0; (d, \sigma) \mapsto (\sigma, d \varphi^{-r_\sigma}). \quad (2.1)$$

By this isomorphism $\mathcal{O}_D^\times \rtimes W_K$ acts on $\coprod_{(\zeta, \zeta') \in \mathcal{S}} \overline{\mathbf{X}}_{\zeta, \zeta'}^c$. We will describe this action.

For $d \in \mathcal{O}_D^\times$, we put $\kappa_1(d) = \bar{d}_1$ and $\kappa_2(d) = -\bar{d}_2/\bar{d}_1^q$, where $d = d_1 + \varphi d_2$ with $d_1 \in \mathcal{O}_{K_2}^\times$ and $d_2 \in \mathcal{O}_{K_2}$. We take $(\zeta, \zeta') \in \mathcal{S}$. We put $f_d = \text{Tr}_{k_2/\mathbb{F}_2}(\zeta^{1-q} \zeta'^{-2} \kappa_2(d))$ for $d \in \mathcal{O}_D^\times$. We briefly recall the definition of $\zeta_{3, \sigma}$, ν_σ and μ_σ for $\sigma \in W_K$ from [IT, Section 6.2.2]. Consult there for detailed discussions. Let $\hat{\zeta}' \in \mu_{q-1}(K)$ be the lift of ζ' . We choose $\zeta'' \in \mu_{3(q-1)}(K^{\text{ur}})$ such that $\zeta''^3 = \hat{\zeta}'^4$. We take $\delta \in K^{\text{ac}}$ such that $\delta^4 - \delta = 1/(\zeta'' \varpi^{1/3})$ and $\delta \varpi^{1/12} \equiv \hat{\zeta}' \zeta''^{-1} \pmod{0}$. We take $\theta \in K^{\text{ac}}$ such that $\theta^2 - \theta = \delta^3$. Note that $v(\delta) = -1/12$, $v(\theta) = -1/8$ and $\delta \in K(\zeta'' \varpi^{1/3}, \theta)$.

Let $\sigma \in W_K$ in this paragraph. We put $\zeta_{3, \sigma} = \sigma(\zeta'' \varpi^{1/3})/(\zeta'' \varpi^{1/3}) \in \mu_3(K^{\text{ur}})$. We take $\nu_\sigma \in \mu_3(K^{\text{ur}}) \cup \{0\}$ such that $\sigma(\delta) \equiv \zeta_{3, \sigma}^{-1}(\delta + \nu_\sigma) \pmod{5/6}$. We choose $\zeta_3 \in \mu_3(K^{\text{ur}})$ such that $\zeta_3 \neq 1$. Then, we can take $\mu_\sigma \in \mu_3(K^{\text{ur}}) \cup \{0\}$ such that $\mu_\sigma \equiv \sigma(\theta) - \theta + \nu_\sigma^2 \delta + \nu_\sigma^3 + \sigma(\zeta_3) - \zeta_3 \pmod{0+}$.

We put $\lambda_\sigma = \sigma(\varpi^{1/(2(q-1))})/\varpi^{1/(2(q-1))} \in \mu_{2(q-1)}(K^{\text{ac}})$ for $\sigma \in W_K$. We define a character $\lambda: W_K \rightarrow k^\times$ by $\lambda(\sigma) = \bar{\lambda}_\sigma$. We put

$$Q = \left\{ g(\alpha, \beta, \gamma) = \begin{pmatrix} \alpha & \beta & \gamma \\ & \alpha^2 & \beta^2 \\ & & \alpha \end{pmatrix} \in GL_3(\mathbb{F}_4) \mid \alpha \gamma^2 + \alpha^2 \gamma = \beta^3 \right\}.$$

We note that $|Q| = 24$. Let $Q \rtimes \mathbb{Z}$ be a semidirect product, where $r \in \mathbb{Z}$ acts on Q by $g(\alpha, \beta, \gamma) \mapsto g(\alpha^{q^r}, \beta^{q^r}, \gamma^{q^r})$. Let $k_2^\times \rtimes \text{Gal}(k_2/k)$ be a semidirect product with a natural action of $\text{Gal}(k_2/k)$ on k_2^\times . We consider \mathbb{F}_4 as a subfield of $k_2 \subset k^{\text{ac}}$. Let Fr_q be the q -th power Frobenius map on k^{ac} .

Then $(Q \rtimes \mathbb{Z}) \times (k_2^\times \rtimes \text{Gal}(k_2/k))$ acts on $\coprod_{(\zeta, \zeta') \in \mathcal{S}} \overline{\mathbf{X}}_{\zeta, \zeta'}^c$ as a scheme over k , where the action of $((g(\alpha, \beta, \gamma), r), (a, \text{Fr}_q^b)) \in (Q \rtimes \mathbb{Z}) \times (k_2^\times \rtimes \text{Gal}(k_2/k))$ induces the isomorphism

$$\overline{\mathbf{X}}_{\zeta, \zeta'} \rightarrow \overline{\mathbf{X}}_{a\zeta^{q^b}, \zeta'}; (z, w) \mapsto (z^{q^{-r}} + \alpha^{-1}\beta w^{q^{-r}} + \alpha^{-1}\gamma, \alpha(w^{q^{-r}} + (\alpha^{-1}\beta)^2)),$$

where we describe a bijection on k^{ac} -valued points.

Proposition 2.1. *The action of $\mathcal{O}_D^\times \rtimes W_K$ on $\coprod_{(\zeta, \zeta') \in \mathcal{S}} \overline{\mathbf{X}}_{\zeta, \zeta'}^c$ gives the homomorphism*

$$\begin{aligned} \Xi_{\zeta'}: \mathcal{O}_D^\times \rtimes W_K &\rightarrow (Q \rtimes \mathbb{Z}) \times (k_2^\times \rtimes \text{Gal}(k_2/k)); \\ (d, \sigma) &\mapsto \left((g(1, 0, f_d)g(\bar{\zeta}_{3, \sigma}, \bar{\zeta}_{3, \sigma}^2 \bar{\nu}_\sigma^2, \bar{\zeta}_{3, \sigma} \bar{\mu}_\sigma), r_\sigma), (\kappa_1(d)\bar{\lambda}_\sigma, \text{Fr}_q^{-r_\sigma}) \right). \end{aligned}$$

Proof. This follows from [IT, Proposition 5.4 and Proposition 6.12]. \square

Let $\Theta_{\zeta'}: W_K \rightarrow Q \rtimes \mathbb{Z}$ be the composite of $\Xi_{\zeta'}|_{W_K}$ with the projection to $Q \rtimes \mathbb{Z}$. By [IT, Proposition 6.13], the map $\Theta_{\zeta'}$ gives an isomorphism $W(K^{\text{ur}}(\varpi^{1/3}, \theta)/K) \simeq Q \rtimes \mathbb{Z}$ and a finite extension of K inside $K^{\text{ur}}(\varpi^{1/3}, \theta)$ corresponds to a finite index subgroup of $Q \rtimes \mathbb{Z}$.

3 Cohomology of elliptic curve

Let ℓ be an odd prime number. We fix an algebraic closure $\overline{\mathbb{Q}}_\ell$ of \mathbb{Q}_ℓ . In the sequel, we consider representations of groups over $\overline{\mathbb{Q}}_\ell$. We put $Q_8 = \{g(1, \beta, \gamma) \in Q\}$, which is a normal subgroup of Q of order 8. Let $C_4 \subset Q_8$ be the cyclic subgroup of order 4 generated by $g(1, 1, \gamma)$ for $\gamma \in \mathbb{F}_4^\times \setminus \{1\}$. Let $Z \subset C_4$ be the subgroup consisting of $g(1, 0, \gamma)$ with $\gamma^2 + \gamma = 0$, which is the center of Q . We take a faithful character ϕ of C_4 . By [BH, 22.2 Lemma], there exists a unique irreducible two-dimensional representation τ of Q such that

$$\tau|_Z \simeq (\phi|_Z)^{\oplus 2}, \quad \text{Tr } \tau(g(\alpha, 0, 0)) = -1 \quad (3.1)$$

for $\alpha \in \mathbb{F}_4^\times \setminus \{1\}$. Let $C_3 \subset Q$ be the cyclic subgroup of order 3 consisting of $g(\alpha, 0, 0)$ with $\alpha \in \mathbb{F}_4^\times$. Then we have

$$\tau = \text{Ind}_{C_4}^Q \phi - \text{Ind}_{Z \times C_3}^Q (\phi|_Z \otimes 1_{C_3}) \quad (3.2)$$

by [BH, 16.4 Lemma 2.(4)] and a proof of [BH, 22.2 Lemma].

Let f be the degree of the extension k over \mathbb{F}_2 . We choose $(-2)^{1/2} \in \overline{\mathbb{Q}}_\ell$. We define a two-dimensional representation τ'_q of $Q \times 2\mathbb{Z} \subset Q \rtimes \mathbb{Z}$ so that $\tau'_q|_Q \simeq \tau$ and $(g(1, 0, 0), 2) \in Q \times 2\mathbb{Z}$ acts by a scalar multiplication $(-2)^{-f}$. In the following, we will define a two-dimensional representation τ_q of $Q \rtimes \mathbb{Z}$ such that $\tau_q|_{Q \times 2\mathbb{Z}} \simeq \tau'_q$. We define a character ϕ_0 of $Q \rtimes \mathbb{Z}$ by sending (g, n) to $(-2)^{fn/2} q^{-n}$.

First, we consider the case where f is even. Then $Q \rtimes \mathbb{Z}$ is the direct product $Q \times \mathbb{Z}$. We define τ_q so that $(g(1, 0, 0), 1) \in Q \times \mathbb{Z}$ acts by a scalar multiplication $(-2)^{-f/2}$. We define a character ϕ_1 of $C_4 \times \mathbb{Z}$ by $\phi_1(g, n) = \phi(g)\phi_0(g, n)$. Then we have $\tau_q|_{Q_8 \times \mathbb{Z}} \simeq \text{Ind}_{C_4 \times \mathbb{Z}}^{Q_8 \times \mathbb{Z}} \phi_1$.

Next, we consider the case where f is odd. We assume that f is odd, until the end of Lemma 3.3. Let $C \subset Q_8 \rtimes \mathbb{Z}$ be the subgroup which consists of $(g(1, \beta, \gamma), n)$ satisfying $g(1, \beta, \gamma) \in C_4$ if n is even, and $g(1, \beta, \gamma) \notin C_4$ if n is odd. We note that the index of C in $Q_8 \rtimes \mathbb{Z}$ is two. We take $\eta \in \overline{\mathbb{Q}}_\ell$ satisfying $\eta^2 + (-2)^{(f+1)/2}\eta + q = 0$. We note that $\eta^4 = -q^2$. We define a character ϕ_1 of C by sending $(g(1, \bar{\zeta}_3, \bar{\zeta}_3), 1)$ to ηq^{-1} and $(g(1, 0, 0), 2)$ to $-q^{-1}$. We note that $(g(1, \bar{\zeta}_3, \bar{\zeta}_3), 1)$ and $(g(1, 0, 0), 2)$ generates C as a group.

We define a character ϕ_2 of $(Z \times C_3) \rtimes \mathbb{Z} \subset Q \rtimes \mathbb{Z}$ so that $\phi_2|_{Z \times C_3} = \phi|_Z \otimes 1_{C_3}$ and ϕ_2 sends $(g(1, 0, 0), 1)$ to $(-2)^{f/2} q^{-1}$. We put $C_6 = Z \times C_3$.

Lemma 3.1. *The representation $\text{Ind}_{C_6 \rtimes \mathbb{Z}}^{Q \rtimes \mathbb{Z}} \phi_2$ is irreducible, and there is a surjective homomorphism*

$$\Psi: \text{Ind}_C^{Q \rtimes \mathbb{Z}} \phi_1 \longrightarrow \text{Ind}_{C_6 \rtimes \mathbb{Z}}^{Q \rtimes \mathbb{Z}} \phi_2.$$

Proof. First we note that $\text{Ind}_C^{Q \rtimes \mathbb{Z}} \phi_1$ and $\text{Ind}_{C_6 \rtimes \mathbb{Z}}^{Q \rtimes \mathbb{Z}} \phi_2$ are semisimple representations, because their twists by ϕ_0^{-1} factor through representations of the finite group $Q \rtimes \mathbb{Z}/2\mathbb{Z}$.

For the first statement, it suffices to show $\dim \text{Hom}_{Q \rtimes \mathbb{Z}}(\text{Ind}_{C_6 \rtimes \mathbb{Z}}^{Q \rtimes \mathbb{Z}} \phi_2, \text{Ind}_{C_6 \rtimes \mathbb{Z}}^{Q \rtimes \mathbb{Z}} \phi_2) = 1$. We have

$$\text{Hom}_{Q \rtimes \mathbb{Z}}(\text{Ind}_{C_6 \rtimes \mathbb{Z}}^{Q \rtimes \mathbb{Z}} \phi_2, \text{Ind}_{C_6 \rtimes \mathbb{Z}}^{Q \rtimes \mathbb{Z}} \phi_2) \simeq \text{Hom}_{C_6 \rtimes \mathbb{Z}}(\phi_2, \text{Res}_{C_6 \rtimes \mathbb{Z}}^{Q \rtimes \mathbb{Z}} \text{Ind}_{C_6 \rtimes \mathbb{Z}}^{Q \rtimes \mathbb{Z}} \phi_2) \quad (3.3)$$

by the Frobenius reciprocity. We have

$$\begin{aligned} (C_6 \rtimes \mathbb{Z}) \backslash (Q \rtimes \mathbb{Z}) / (C_6 \rtimes \mathbb{Z}) &= \{[(g(1, 0, 0), 0)], [(g(1, 1, \bar{\zeta}_3), 0)]\} \\ (C_6 \rtimes \mathbb{Z}) \cap ((g(1, 1, \bar{\zeta}_3), 0)(C_6 \rtimes \mathbb{Z})(g(1, 1, \bar{\zeta}_3), 0)^{-1}) &= Z \rtimes \mathbb{Z} \subset Q \rtimes \mathbb{Z} \end{aligned}$$

where $[\cdot]$ denotes an equivalent class. Hence, we obtain

$$\text{Res}_{C_6 \rtimes \mathbb{Z}}^{Q \rtimes \mathbb{Z}} \text{Ind}_{C_6 \rtimes \mathbb{Z}}^{Q \rtimes \mathbb{Z}} \phi_2 \simeq \phi_2 \oplus \text{Ind}_{Z \rtimes \mathbb{Z}}^{C_6 \rtimes \mathbb{Z}} \phi'_2 \quad (3.4)$$

where $\phi'_2((g, n)) = \phi_2((g(1, 1, \bar{\zeta}_3), 0)^{-1}(g, n)(g(1, 1, \bar{\zeta}_3), 0))$ for $(g, n) \in Z \rtimes \mathbb{Z}$. We have

$$\text{Hom}_{C_6 \rtimes \mathbb{Z}}(\phi_2, \text{Ind}_{Z \rtimes \mathbb{Z}}^{C_6 \rtimes \mathbb{Z}} \phi'_2) \simeq \text{Hom}_{Z \rtimes \mathbb{Z}}(\phi_2|_{Z \rtimes \mathbb{Z}}, \phi'_2) = 0 \quad (3.5)$$

by the Frobenius reciprocity. By (3.3), (3.4) and (3.5), we have

$$\dim \text{Hom}_{Q \rtimes \mathbb{Z}}(\text{Ind}_{C_6 \rtimes \mathbb{Z}}^{Q \rtimes \mathbb{Z}} \phi_2, \text{Ind}_{C_6 \rtimes \mathbb{Z}}^{Q \rtimes \mathbb{Z}} \phi_2) = 1.$$

For the second statement, it suffices to show $\dim \text{Hom}_{Q \rtimes \mathbb{Z}}(\text{Ind}_C^{Q \rtimes \mathbb{Z}} \phi_1, \text{Ind}_{C_6 \rtimes \mathbb{Z}}^{Q \rtimes \mathbb{Z}} \phi_2) = 1$. This follows from

$$C \backslash (Q \rtimes \mathbb{Z}) / (C_6 \rtimes \mathbb{Z}) = \{[(g(1, 0, 0), 0)]\}$$

by the similar arguments as above. \square

We define τ_q by the kernel of Ψ . Then we have $\tau_q|_{Q \rtimes 2\mathbb{Z}} \simeq \tau'_q$ by the definition of τ'_q and (3.2). In particular, τ_q is irreducible.

Lemma 3.2. *We have an isomorphism $\tau_q|_{Q_8 \rtimes \mathbb{Z}} \simeq \text{Ind}_{C_8 \rtimes \mathbb{Z}}^{Q_8 \rtimes \mathbb{Z}} \phi_1$.*

Proof. By Lemma 3.1, we see that $\text{Ind}_C^{Q \rtimes \mathbb{Z}} \phi_1$ is a direct sum of a two-dimensional irreducible representation and a four-dimensional irreducible representation. Hence, we see that $\text{Ind}_C^{Q_8 \rtimes \mathbb{Z}} \phi_1$ is irreducible. We have

$$\begin{aligned} \dim \text{Hom}_{Q_8 \rtimes \mathbb{Z}}(\text{Ind}_C^{Q_8 \rtimes \mathbb{Z}} \phi_1, \text{Res}_{Q_8 \rtimes \mathbb{Z}}^{Q \rtimes \mathbb{Z}} \text{Ind}_C^{Q \rtimes \mathbb{Z}} \phi_1) &= \dim \text{Hom}_{Q \rtimes \mathbb{Z}}(\text{Ind}_C^{Q \rtimes \mathbb{Z}} \phi_1, \text{Ind}_C^{Q \rtimes \mathbb{Z}} \phi_1) = 2 \\ \dim \text{Hom}_{Q_8 \rtimes \mathbb{Z}}(\text{Ind}_C^{Q_8 \rtimes \mathbb{Z}} \phi_1, \text{Res}_{Q_8 \rtimes \mathbb{Z}}^{Q \rtimes \mathbb{Z}} \text{Ind}_{C_6 \rtimes \mathbb{Z}}^{Q \rtimes \mathbb{Z}} \phi_2) &= \dim \text{Hom}_{Q \rtimes \mathbb{Z}}(\text{Ind}_C^{Q \rtimes \mathbb{Z}} \phi_1, \text{Ind}_{C_6 \rtimes \mathbb{Z}}^{Q \rtimes \mathbb{Z}} \phi_2) = 1 \end{aligned}$$

by the proof of Lemma 3.1. Therefore, we obtain the claim by the definition of τ_q . \square

For a representation π of a group G on V and $g \in G$, the trace of g on V is denoted by $\text{tr}(g; \pi)$ or $\text{tr}(g; V)$, and the determinant of g on V is denoted by $\det(g; \pi)$ or $\det(g; V)$.

Lemma 3.3. *The representation τ_q is the unique representation satisfying $\tau_q|_{Q \rtimes 2\mathbb{Z}} \simeq \tau'_q$ and*

$$\text{tr}((g(1, \bar{\zeta}_3, \bar{\zeta}_3), 1); \tau_q) = -(-2)^{\frac{\ell+1}{2}} q^{-1}. \quad (3.6)$$

Proof. There are two representations of $Q \rtimes \mathbb{Z}$ extending τ'_q , and one is a twist of another by the character $Q \rtimes \mathbb{Z} \rightarrow \overline{\mathbb{Q}}_\ell^\times; (g, n) \mapsto (-1)^n$. Therefore, the uniqueness in the statement follows.

It remains to show that τ_q satisfies (3.6). Let C' be the subgroup of C generated by $(g(1, \bar{\zeta}_3, \bar{\zeta}_3), 1)$, and C'' be the subgroup of C generated by $(g(1, 0, 1), 4)$. We have $C'' \subset C'$, because $(g(1, 0, 1), 4) = (g(1, \bar{\zeta}_3, \bar{\zeta}_3), 1)^4$.

Then we have

$$\begin{aligned} C' \backslash (Q \rtimes \mathbb{Z}) / C &= \{[(g(1, 0, 0), 0)], [(g(1, \bar{\zeta}_3, \bar{\zeta}_3), 0)], [(g(\bar{\zeta}_3, 0, 0), 0)]\}, \\ C' \cap ((g(\bar{\zeta}_3, 0, 0), 0)C(g(\bar{\zeta}_3, 0, 0), 0)^{-1}) &= C'' \subset Q \rtimes \mathbb{Z}. \end{aligned}$$

Hence, we obtain

$$\text{Res}_{C'}^{Q \rtimes \mathbb{Z}} \text{Ind}_C^{Q \rtimes \mathbb{Z}} \phi_1 \simeq \phi_1|_{C'} \oplus \phi'_1 \oplus \text{Ind}_{C''}^{C'}(\phi_1|_{C''}) \quad (3.7)$$

where $\phi'_1((g, n)) = \phi_1((g(1, \bar{\zeta}_3, \bar{\zeta}_3), 0)^{-1}(g, n)(g(1, \bar{\zeta}_3, \bar{\zeta}_3), 0))$ for $(g, n) \in C'$. Therefore we have

$$\text{tr}((g(1, \bar{\zeta}_3, \bar{\zeta}_3), 1); \text{Ind}_C^{Q \rtimes \mathbb{Z}} \phi_1) = \frac{\eta}{q} - \frac{\eta^3}{q^2} = \frac{-(-2)^{\frac{f+1}{2}}}{q} \quad (3.8)$$

using

$$(g(1, \bar{\zeta}_3, \bar{\zeta}_3), 0)^{-1}(g(1, \bar{\zeta}_3, \bar{\zeta}_3), 1)(g(1, \bar{\zeta}_3, \bar{\zeta}_3), 0) = (g(1, \bar{\zeta}_3, \bar{\zeta}_3), 1)^3(g(1, 0, 0), -2).$$

By $C' \backslash (Q \rtimes \mathbb{Z}) / (C_6 \rtimes \mathbb{Z}) = \{[(g(1, 0, 0), 0)]\}$ and $C' \cap (C_6 \rtimes \mathbb{Z}) = C''$, we obtain

$$\text{Res}_{C'}^{Q \rtimes \mathbb{Z}} \text{Ind}_{C_6 \rtimes \mathbb{Z}}^{Q \rtimes \mathbb{Z}} \phi_2 \simeq \text{Ind}_{C''}^{C'}(\phi_2|_{C''}). \quad (3.9)$$

Therefore we have

$$\text{tr}((g(1, \bar{\zeta}_3, \bar{\zeta}_3), 1); \text{Ind}_{C_6 \rtimes \mathbb{Z}}^{Q \rtimes \mathbb{Z}} \phi_2) = 0. \quad (3.10)$$

The equation (3.6) follows from (3.8) and (3.10). \square

Lemma 3.4. *We have $\det((g, n); \tau_q) = q^{-n}$ for $(g, n) \in Q \rtimes \mathbb{Z}$.*

Proof. First we are going to show that $\det \tau = 1$. We see that $\det \tau$ factors through Q/Q_8 , because Q/Q_8 is the maximal abelian quotient of Q . By (3.2), we know that τ is self-dual. Hence, the character of Q/Q_8 induced from $\det \tau$ is trivial. Therefore, we have $\det \tau = 1$. Then we see that $\det \tau_q$ factors through \mathbb{Z} , because $\det \tau = 1$.

First, we assume that f is even. Then we have $\det((g(1, 0, 0), 1); \tau_q) = q^{-1}$, and this shows the claim.

Next, we assume that f is odd. Then, it suffices to show that $\det((g(1, \bar{\zeta}_3, \bar{\zeta}_3), 1); \tau_q) = q^{-1}$. This follows from (3.7) and (3.9), because Ψ is surjective. \square

Let \mathcal{E} be the elliptic curve over \mathbb{F}_2 defined by $z^2 + z = w^3$. Then $(g(\alpha, \beta, \gamma), r) \in Q \rtimes \mathbb{Z}$ acts on $\mathcal{E}_{k^{\text{ac}}}$ by $(z, w) \mapsto (z^{q^{-r}} + \alpha^{-1}\beta w^{q^{-r}} + \alpha^{-1}\gamma, \alpha(w^{q^{-r}} + (\alpha^{-1}\beta)^2))$. The action of $Q \rtimes \mathbb{Z}$ gives a representation $H^1(\mathcal{E}_{k^{\text{ac}}}, \overline{\mathbb{Q}}_\ell)$ of $Q \rtimes \mathbb{Z}$ by the pullback by the inverse. For a representation V of $Q \rtimes \mathbb{Z}$ and an integer m , we write $V(m)$ for the twist of V by the character $Q \rtimes \mathbb{Z} \ni (g, n) \mapsto q^{-mn}$.

Proposition 3.5. *We have an isomorphism $H^1(\mathcal{E}_{k^{\text{ac}}}, \overline{\mathbb{Q}}_\ell)(1) \simeq \tau_q$ as representations of $Q \rtimes \mathbb{Z}$.*

Proof. Let fr_{2^m} be the 2^m -th power geometric Frobenius map of $\mathcal{E}_{k^{\text{ac}}}$ for any positive integer m . Then we have $\text{tr}(\text{fr}_2^*; H^1(\mathcal{E}_{k^{\text{ac}}}, \overline{\mathbb{Q}}_\ell)) = 0$ and $\text{tr}(\text{fr}_4^*; H^1(\mathcal{E}_{k^{\text{ac}}}, \overline{\mathbb{Q}}_\ell)) = -4$ by $|\mathcal{E}(\mathbb{F}_2)| = 3$ and $|\mathcal{E}(\mathbb{F}_4)| = 9$. Hence we obtain $\text{tr}(\text{fr}_q^*; H^1(\mathcal{E}_{k^{\text{ac}}}, \overline{\mathbb{Q}}_\ell)) = ((-2)^{1/2})^f + (-(-2)^{1/2})^f$.

As Q -representations, $H^1(\mathcal{E}_{k^{\text{ac}}}, \overline{\mathbb{Q}}_\ell) \simeq \tau$ by [IT, Lemma 7.7]. Then we have $H^1(\mathcal{E}_{k^{\text{ac}}}, \overline{\mathbb{Q}}_\ell)(1) \simeq \tau'_q$ as representations of $Q \times 2\mathbb{Z}$, because

$$\begin{aligned} \text{tr}((g(1, 0, 0), 2); H^1(\mathcal{E}_{k^{\text{ac}}}, \overline{\mathbb{Q}}_\ell)(1)) &= \text{tr}((g(1, 0, 0), -2)^*; H^1(\mathcal{E}_{k^{\text{ac}}}, \overline{\mathbb{Q}}_\ell))q^{-2} \\ &= \text{tr}(\text{fr}_{q^2}^*; H^1(\mathcal{E}_{k^{\text{ac}}}, \overline{\mathbb{Q}}_\ell))q^{-2} = 2(-2)^{-f} \end{aligned}$$

for $(g(1, 0, 0), 2) \in Q \times \mathbb{Z}$.

First, we consider the case where f is even. Then $Z \times \mathbb{Z}$ acts on $H^1(\mathcal{E}_{k^{\text{ac}}}, \overline{\mathbb{Q}}_\ell)(1)$ by scalar multiplications. Hence we have $H^1(\mathcal{E}_{k^{\text{ac}}}, \overline{\mathbb{Q}}_\ell)(1) \simeq \tau_q$, because

$$\text{tr}((g(1, 0, 0), 1); H^1(\mathcal{E}_{k^{\text{ac}}}, \overline{\mathbb{Q}}_\ell)(1)) = \text{tr}(\text{fr}_q^*; H^1(\mathcal{E}_{k^{\text{ac}}}, \overline{\mathbb{Q}}_\ell))q^{-1} = 2(-2)^{-f/2}$$

for $(g(1, 0, 0), 1) \in Q \times \mathbb{Z}$.

Next, we consider the case where f is odd. It suffices to consider the case where $f = 1$, because the claimed isomorphisms for general cases are induced from the isomorphism for this case by the group homomorphism $Q \rtimes \mathbb{Z} \rightarrow Q \rtimes \mathbb{Z}; (g, n) \mapsto (g, fn)$.

We assume that $f = 1$. It suffices to show that $H^1(\mathcal{E}_{k^{\text{ac}}}, \overline{\mathbb{Q}}_\ell)(1)$ satisfies the conditions in Lemma 3.3. We have already checked the first condition. Let $\text{Fr}_{2, \mathcal{E}}$ be the absolute 2-th power Frobenius map on $\mathcal{E}_{k^{\text{ac}}}$. By the Lefschetz trace formula, we have

$$\begin{aligned} 2 + 1 - \text{tr}((g(1, \bar{\zeta}_3, \bar{\zeta}_3), 1); H^1(\mathcal{E}_{k^{\text{ac}}}, \overline{\mathbb{Q}}_\ell)) &= |\{P \in \mathcal{E}(k^{\text{ac}}) \mid ((g(1, \bar{\zeta}_3, \bar{\zeta}_3), 1)^{-1} \circ \text{Fr}_{2, \mathcal{E}})P = P\}| \\ &= |\{(z, w) \in k^{\text{ac}} \times k^{\text{ac}} \mid z^2 + z = w^3, z = z^2 + \bar{\zeta}_3^2 w^2 + \bar{\zeta}_3, w = w^2 + \bar{\zeta}_3\}| + 1 \\ &= |\{(z, w) \in k^{\text{ac}} \times k^{\text{ac}} \mid z^2 + z = w^3, w^3 + \bar{\zeta}_3^2 w^2 + \bar{\zeta}_3 = 0, w^2 + w + \bar{\zeta}_3 = 0\}| + 1 = 1. \end{aligned}$$

Therefore, the condition (3.6) is satisfied for $H^1(\mathcal{E}_{k^{\text{ac}}}, \overline{\mathbb{Q}}_\ell)(1)$. \square

Let $\tau_{\zeta'}$ be the representation of W_K induced from the $(Q \rtimes \mathbb{Z})$ -representation τ_q by $\Theta_{\zeta'}$. We say that a continuous two-dimensional representation V of W_K over $\overline{\mathbb{Q}}_\ell$ is primitive, if there is no pair of a quadratic extension K' and a continuous character χ of $W_{K'}$ such that $V \simeq \text{Ind}_{W_{K'}}^{W_K} \chi$. The representation $\tau_{\zeta'}$ is primitive of Artin conductor 3 by [IT, Lemma 7.8].

4 Realization of correspondence

Let $\chi_2: \mathbb{F}_2 \rightarrow \overline{\mathbb{Q}}_\ell^\times$ be the non-trivial character. We put

$$\mathfrak{J} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathcal{O}_K) \mid c \equiv 0 \pmod{\mathfrak{p}} \right\}, \quad \mathfrak{P} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathfrak{J} \mid a \equiv d \equiv 0 \pmod{\mathfrak{p}} \right\}, \quad (4.1)$$

and $U_{\mathfrak{J}}^1 = 1 + \mathfrak{P} \subset M_2(\mathcal{O}_K)$. We put $L = K(\varphi) \subset D$, and consider L as a K -subalgebra of $M_2(K)$ by the embedding

$$L \hookrightarrow M_2(K); \varphi \mapsto \begin{pmatrix} 0 & 1 \\ \varpi & 0 \end{pmatrix}. \quad (4.2)$$

We put $U_D^1 = 1 + \varphi \mathcal{O}_D$. We take an additive character $\psi_K: K \rightarrow \overline{\mathbb{Q}}_\ell^\times$ such that $\psi_K(a) = (\chi_2 \circ \text{Tr}_{k/\mathbb{F}_2})(\bar{a})$ for $a \in \mathcal{O}_K$.

For a finite abelian group A , the character group $\text{Hom}_{\mathbb{Z}}(A, \overline{\mathbb{Q}}_\ell^\times)$ is denoted by A^\vee . Let $\zeta' \in k^\times$, $\chi \in (k^\times)^\vee$ and $c \in \overline{\mathbb{Q}}_\ell^\times$. We define a character $\Lambda_{\zeta', \chi, c}: L^\times U_{\mathfrak{J}}^1 \rightarrow \overline{\mathbb{Q}}_\ell^\times$ by $\Lambda_{\zeta', \chi, c}(\varphi) = -c$, $\Lambda_{\zeta', \chi, c}(a) = \chi(\bar{a})$ for $a \in \mathcal{O}_L^\times$ and

$$\Lambda_{\zeta', \chi, c}(x) = (\psi_K \circ \text{tr})(\hat{\zeta}'^{-2} \varphi^{-1}(x - 1))$$

for $x \in U_{\mathfrak{J}}^1$. We put $\pi_{\zeta', \chi, c} = \text{c-Ind}_{L^\times U_{\mathfrak{J}}^1}^{GL_2(K)} \Lambda_{\zeta', \chi, c}$. Next, we define a character $\theta_{\zeta', \chi, c}: L^\times U_D^1 \rightarrow \overline{\mathbb{Q}}_\ell^\times$ by $\theta_{\zeta', \chi, c}(\varphi) = c$, $\theta_{\zeta', \chi, c}(a) = \chi(\bar{a})$ for $a \in \mathcal{O}_L^\times$ and $\theta_{\zeta', \chi, c}(d) = (\chi_2 \circ \text{Tr}_{k_2/\mathbb{F}_2})(\zeta'^{-2} \kappa_2(d))$ for $d \in U_D^1$. We put $\rho_{\zeta', \chi, c} = \text{Ind}_{L^\times U_D^1}^{D^\times} \theta_{\zeta', \chi, c}$.

Proposition 4.1. *For $\zeta' \in k^\times$, $\chi \in (k^\times)^\vee$ and $c \in \overline{\mathbb{Q}}_\ell^\times$, we have $\text{JL}(\rho_{\zeta', \chi, c}) = \pi_{\zeta', \chi, c}$.*

Proof. This follows from [BH, 56.5], because

$$(\psi_K \circ \text{Tr}_{D/K})(\hat{\zeta}'^{-2} \varphi^{-1}(d - 1)) = (\chi_2 \circ \text{Tr}_{k_2/\mathbb{F}_2})(\zeta'^{-2} \kappa_2(d))$$

for $d \in U_D^1$. \square

For $c \in \overline{\mathbb{Q}}_\ell^\times$, let $\phi_c: W_K \rightarrow \overline{\mathbb{Q}}_\ell^\times$ be the character defined by $\phi_c(\sigma) = c^{r_\sigma}$. For $\zeta' \in k^\times$, $\chi \in (k^\times)^\vee$ and $c \in \overline{\mathbb{Q}}_\ell^\times$, we put $\tau_{\zeta', \chi, c} = \tau_{\zeta'} \otimes (\chi \circ \lambda) \otimes \phi_c$. For a representation V of a Weil group and an integer m , we write $V(m)$ for the m -times Tate twist of V .

Theorem 4.2. For $\zeta' \in k^\times$, $\chi \in (k^\times)^\vee$ and $c \in \overline{\mathbb{Q}}_\ell^\times$, we have

$$\mathrm{Hom}_{GL_2(K)}(H_{LT}^1, \pi_{\zeta', \chi, c}) \simeq \tau_{\zeta', \chi, c} \otimes \rho_{\zeta', \chi, c}$$

as representations of $W_K \times D^\times$.

Proof. Let $\zeta' \in k^\times$, $\chi \in (k^\times)^\vee$ and $c \in \overline{\mathbb{Q}}_\ell^\times$. By Proposition 1.1 and Proposition 4.1, we know that

$$\mathrm{Hom}_{D^\times}(\rho_{\zeta', \chi, c}, \mathrm{Hom}_{GL_2(K)}(H_{LT}^1, \pi_{\zeta', \chi, c})) \simeq \tau'$$

for some two-dimensional W_K -representation τ' . First, we will show that $\tau' = \tau_{\zeta', \chi, c}$.

We put $H_{\mathbf{X}}^1 = \varinjlim_n H_c^1(\mathbf{X}(\mathfrak{p}^n)_{\widehat{K}^{\mathrm{ac}}}, \overline{\mathbb{Q}}_\ell)$ and

$$(GL_2(K) \times W_K \times D^\times)^0 = \{(g, \sigma, d) \in GL_2(K) \times W_K \times D^\times \mid |\det(g)^{-1} \mathrm{Nrd}_{D/K}(d)|_K \cdot |\sigma| = 1\}.$$

Then we have

$$\begin{aligned} \mathrm{Hom}_{GL_2(K)}(H_{LT}^1, \pi_{\zeta', \chi, c}) &\simeq \mathrm{Hom}_{GL_2(K)}(\mathrm{c}\text{-Ind}_{(GL_2(K) \times W_K \times D^\times)^0}^{GL_2(K) \times W_K \times D^\times} H_{\mathbf{X}}^1, \pi_{\zeta', \chi, c}) \\ &\subset \mathrm{Hom}_{K_1(\mathfrak{p}^3)}(\mathrm{c}\text{-Ind}_{K_1(\mathfrak{p}^3) \times (W_K \times D^\times)^0}^{K_1(\mathfrak{p}^3) \times W_K \times D^\times} H_{\mathbf{X}}^1, \pi_{\zeta', \chi, c}) \\ &\subset \mathrm{Hom}_{\overline{\mathbb{Q}}_\ell}(\mathrm{c}\text{-Ind}_{(W_K \times D^\times)^0}^{W_K \times D^\times} H_c^1(\mathbf{X}_1(\mathfrak{p}^3)_{\widehat{K}^{\mathrm{ac}}}, \overline{\mathbb{Q}}_\ell), \overline{\mathbb{Q}}_\ell), \end{aligned}$$

where the last inclusion follows by taking the $K_1(\mathfrak{p}^3)$ -invariant part, because the conductor of $\pi_{\zeta', \chi, c}$ is three. Hence, we obtain

$$\begin{aligned} \tau' &\simeq \mathrm{Hom}(\rho_{\zeta', \chi, c}, \mathrm{Hom}_{GL_2(K)}(H_{LT}^1, \pi_{\zeta', \chi, c})) \\ &\subset \mathrm{Hom}_{D^\times}(\rho_{\zeta', \chi, c}, (\mathrm{c}\text{-Ind}_{(W_K \times D^\times)^0}^{W_K \times D^\times} H_c^1(\mathbf{X}_1(\mathfrak{p}^3)_{\widehat{K}^{\mathrm{ac}}}, \overline{\mathbb{Q}}_\ell))^*) \\ &\simeq \mathrm{Hom}_{D^\times}(\mathrm{c}\text{-Ind}_{(W_K \times D^\times)^0}^{W_K \times D^\times} H_c^1(\mathbf{X}_1(\mathfrak{p}^3)_{\widehat{K}^{\mathrm{ac}}}, \overline{\mathbb{Q}}_\ell), \rho_{\zeta', \chi, c}^*) \\ &\simeq \mathrm{Hom}_{D^\times}(\mathrm{c}\text{-Ind}_{(W_K \times D^\times)^0}^{W_K \times D^\times} (\bigoplus_{\zeta \in k_2^\times} H^1(\overline{\mathbf{X}}_{\zeta, \zeta'}, \overline{\mathbb{Q}}_\ell)^*(-1)), \rho_{\zeta', \chi, c}^*), \end{aligned} \tag{4.3}$$

where the last isomorphism follows from [IT, Proposition 7.3, Proposition 7.9 and Theorem 7.16] by studying only \mathcal{O}_D^\times -actions. As vector spaces, the last space is isomorphic to

$$\mathrm{Hom}_{D^\times}(\mathrm{c}\text{-Ind}_{\mathcal{O}_D^\times}^{D^\times} (\bigoplus_{\zeta \in k_2^\times} H^1(\overline{\mathbf{X}}_{\zeta, \zeta'}, \overline{\mathbb{Q}}_\ell)^*), \rho_{\zeta', \chi, c}^*) \simeq \mathrm{Hom}_{\mathcal{O}_D^\times}(\bigoplus_{\zeta \in k_2^\times} H^1(\overline{\mathbf{X}}_{\zeta, \zeta'}, \overline{\mathbb{Q}}_\ell)^*, \rho_{\zeta', \chi, c}^*|_{\mathcal{O}_D^\times})$$

which is two-dimensional by [IT, Proposition 7.9]. Hence, the inclusion in (4.3) is an equality. Therefore it suffices to show that there is a non-trivial homomorphism

$$\mathrm{c}\text{-Ind}_{(W_K \times D^\times)^0}^{W_K \times D^\times} (\bigoplus_{\zeta \in k_2^\times} H^1(\overline{\mathbf{X}}_{\zeta, \zeta'}, \overline{\mathbb{Q}}_\ell)^*(-1)) \longrightarrow \tau_{\zeta', \chi, c}^* \otimes \rho_{\zeta', \chi, c}^*$$

as representations of $W_K \times D^\times$. By the Frobenius reciprocity, this is equivalent to give a non-trivial homomorphism

$$(\tau_{\zeta', \chi, c} \otimes \rho_{\zeta', \chi, c})|_{(W_K \times D^\times)^0} \longrightarrow \bigoplus_{\zeta \in k_2^\times} H^1(\overline{\mathbf{X}}_{\zeta, \zeta'}, \overline{\mathbb{Q}}_\ell)(1)$$

as representations of $(W_K \times D^\times)^0$. We put

$$(\mathcal{O}_D^\times \rtimes W_K)^0 = \{(d, \sigma) \in \mathcal{O}_D^\times \rtimes W_K \mid \kappa_1(d) \bar{\lambda}_\sigma = 1\}$$

and consider this group as a subgroup of $(W_K \times D^\times)^0$ by the isomorphism (2.1). Then we have

$$\bigoplus_{\zeta \in k_2^\times} H^1(\overline{\mathbf{X}}_{\zeta, \zeta'}, \overline{\mathbb{Q}}_\ell)(1) \simeq \mathrm{Ind}_{(\mathcal{O}_D^\times \rtimes W_K)^0}^{(W_K \times D^\times)^0} H^1(\overline{\mathbf{X}}_{1, \zeta'}, \overline{\mathbb{Q}}_\ell)(1),$$

because, by Proposition 2.1, the action of $\mathcal{O}_D^\times \rtimes W_K$ on $\coprod_{(\zeta, \zeta') \in \mathcal{S}} \overline{\mathbf{X}}_{\zeta, \zeta'}^c$ permutes the connected components transitively, and $(\mathcal{O}_D^\times \rtimes W_K)^0$ is the stabilizer of the connected component $\overline{\mathbf{X}}_{1, \zeta'}^c$. Hence, we have

$$\begin{aligned} & \text{Hom}_{(W_K \times D^\times)^0}((\tau_{\zeta', \chi, c} \otimes \rho_{\zeta', \chi, c})|_{(W_K \times D^\times)^0}, \bigoplus_{\zeta \in k_2^\times} H^1(\overline{\mathbf{X}}_{\zeta, \zeta'}^c, \overline{\mathbb{Q}}_\ell)(1)) \\ & \simeq \text{Hom}_{(\mathcal{O}_D^\times \rtimes W_K)^0}((\tau_{\zeta', \chi, c} \otimes \rho_{\zeta', \chi, c})|_{(\mathcal{O}_D^\times \rtimes W_K)^0}, H^1(\overline{\mathbf{X}}_{1, \zeta'}^c, \overline{\mathbb{Q}}_\ell)(1)). \end{aligned}$$

Since $\tau_{\zeta', \chi, c} \otimes \rho_{\zeta', \chi, c} \simeq \text{Ind}_{W_K \times L \times U_D^1}^{W_K \times D^\times}(\tau_{\zeta', \chi, c} \otimes \theta_{\zeta', \chi, c})$ and $(\mathcal{O}_D^\times \rtimes W_K)^0 \subset W_K \times L \times U_D^1$, we have a non-trivial homomorphism

$$(\tau_{\zeta', \chi, c} \otimes \rho_{\zeta', \chi, c})|_{(\mathcal{O}_D^\times \rtimes W_K)^0} \longrightarrow (\tau_{\zeta', \chi, c} \otimes \theta_{\zeta', \chi, c})|_{(\mathcal{O}_D^\times \rtimes W_K)^0}$$

by the Frobenius reciprocity. Hence, it suffices to show there is a non-trivial homomorphism

$$(\tau_{\zeta', \chi, c} \otimes \theta_{\zeta', \chi, c})|_{(\mathcal{O}_D^\times \rtimes W_K)^0} \longrightarrow H^1(\overline{\mathbf{X}}_{1, \zeta'}^c, \overline{\mathbb{Q}}_\ell)(1)$$

as representations of $(\mathcal{O}_D^\times \rtimes W_K)^0$. We put $W'_K = \{(\lambda_\sigma^{-1}, \sigma) \in (\mathcal{O}_D^\times \rtimes W_K)^0 \mid \sigma \in W_K\}$. We consider U_D^1 as a subgroup of $(\mathcal{O}_D^\times \rtimes W_K)^0$ by identifying $d \in U_D^1$ with $(d, 1) \in (\mathcal{O}_D^\times \rtimes W_K)^0$. Then we have an isomorphism

$$(\tau_{\zeta', \chi, c} \otimes \theta_{\zeta', \chi, c})|_{W'_K} \longrightarrow H^1(\overline{\mathbf{X}}_{1, \zeta'}^c, \overline{\mathbb{Q}}_\ell)(1)|_{W'_K}$$

as representation of W'_K by Proposition 3.5 and the definition of $\tau_{\zeta', \chi, c}$ and $\theta_{\zeta', \chi, c}$. This isomorphism is compatible with the action of U_D^1 by Proposition 2.1 and by Proposition 3.5. Then this is an isomorphism as representations of $(\mathcal{O}_D^\times \rtimes W_K)^0$, because $(\mathcal{O}_D^\times \rtimes W_K)^0$ is generated by W'_K and U_D^1 . Thus we have proved that

$$\text{Hom}_{D^\times}(\rho_{\zeta', \chi, c}, \text{Hom}_{GL_2(K)}(H_{\text{LT}}^1, \pi_{\zeta', \chi, c})) \simeq \tau_{\zeta', \chi, c}. \quad (4.4)$$

By (4.4), we see that $\text{Hom}_{GL_2(K)}(H_{\text{LT}}^1, \pi_{\zeta', \chi, c})$ is an irreducible representation of $W_K \times D^\times$. Let $\xi_c: D^\times \rightarrow \overline{\mathbb{Q}}_\ell^\times$ be the character defined by $\xi_c(d) = c^{v(\text{Nrd}_{D/K}(d))}$. By (4.4), we have

$$\begin{aligned} & \text{Hom}_{D^\times}(\rho_{\zeta', \chi, c} \otimes \xi_c^{-1}, \text{Hom}_{GL_2(K)}(H_{\text{LT}}^1, \pi_{\zeta', \chi, c}) \otimes \phi_{c(-2)-f/2}^{-1} \otimes (\chi \circ \lambda)^{-1} \otimes \xi_c^{-1}) \\ & \simeq \tau_{\zeta', \chi, c} \otimes \phi_{c(-2)-f/2}^{-1} \otimes (\chi \circ \lambda)^{-1}. \end{aligned}$$

Then we see that $\tau_{\zeta', \chi, c} \otimes \phi_{c(-2)-f/2}^{-1} \otimes (\chi \circ \lambda)^{-1}$ and $\rho_{\zeta', \chi, c} \otimes \xi_c^{-1}$ factor through representations of $Q \rtimes \mathbb{Z}/2\mathbb{Z}$ and $D^\times/(\varpi^\mathbb{Z}(1 + \varpi\mathcal{O}_D))$ respectively. Hence, the $(W_K \times D^\times)$ -representation $\text{Hom}_{GL_2(K)}(H_{\text{LT}}^1, \pi_{\zeta', \chi, c}) \otimes \phi_{c(-2)-f/2}^{-1} \otimes (\chi \circ \lambda)^{-1} \otimes \xi_c^{-1}$ factors through a representation of the finite group $(Q \rtimes \mathbb{Z}/2\mathbb{Z}) \times (D^\times/(\varpi^\mathbb{Z}(1 + \varpi\mathcal{O}_D)))$. Then we have

$$\begin{aligned} & \text{Hom}_{GL_2(K)}(H_{\text{LT}}^1, \pi_{\zeta', \chi, c}) \otimes \phi_{c(-2)-f/2}^{-1} \otimes (\chi \circ \lambda)^{-1} \otimes \xi_c^{-1} \\ & \simeq (\tau_{\zeta', \chi, c} \otimes \phi_{c(-2)-f/2}^{-1} \otimes (\chi \circ \lambda)^{-1}) \otimes (\rho_{\zeta', \chi, c} \otimes \xi_c^{-1}) \end{aligned}$$

because an irreducible representation of a product of two finite groups is isomorphic to a tensor product of irreducible representations of the two groups. Therefore, we have the claim. \square

5 Local Langlands correspondence

In this section, we prove that the correspondence in Theorem 4.2 actually gives the local Langlands correspondence.

We write $\mathcal{G}_2(K, \overline{\mathbb{Q}}_\ell)$ for the set of equivalent classes of two-dimensional semisimple Weil-Deligne representations of W_K over $\overline{\mathbb{Q}}_\ell$, and $\text{Irr}(GL_2(K), \overline{\mathbb{Q}}_\ell)$ for the set of equivalent classes of irreducible

smooth representations of $GL_2(K)$. For $\pi \in \text{Irr}(GL_2(K), \overline{\mathbb{Q}}_\ell)$, let ω_π denote the central character of π . Let $\text{LL}_\ell: \mathcal{G}_2(K, \overline{\mathbb{Q}}_\ell) \rightarrow \text{Irr}(GL_2(K), \overline{\mathbb{Q}}_\ell)$ be the ℓ -adic Langlands correspondence (c.f. [BH, 35.1]). If we take an isomorphism $\iota: \overline{\mathbb{Q}}_\ell \simeq \mathbb{C}$, then ${}^\iota\tau$ and ${}^\iota\pi$ denote the representations over \mathbb{C} associated to τ and π by ι respectively for $\tau \in \mathcal{G}_2(K, \overline{\mathbb{Q}}_\ell)$ and $\pi \in \text{Irr}(GL_2(K), \overline{\mathbb{Q}}_\ell)$. We use similar notations also over a finite extension of K .

Remark 5.1. *The ℓ -adic Langlands correspondence LL_ℓ satisfy that $\omega_{\text{LL}_\ell(\tau)} \circ \text{Art}_K^{-1} = (\det \tau) \otimes |\cdot|^{-1}$ for $\tau \in \mathcal{G}_2(K, \overline{\mathbb{Q}}_\ell)$. If we take an isomorphism $\iota: \overline{\mathbb{Q}}_\ell \simeq \mathbb{C}$, then we have*

$$\varepsilon({}^\iota\tau, s, \psi) = \varepsilon({}^\iota\text{LL}_\ell(\tau), s + 1/2, \psi)$$

for any non-trivial additive character $\psi: K \rightarrow \mathbb{C}^\times$.

For a finite extension K' of K , we define an additive character $\psi_{K'}: K' \rightarrow \overline{\mathbb{Q}}_\ell^\times$ by $\psi_{K'} = \psi_K \circ \text{Tr}_{K'/K}$, and let $v_{K'}$ be the normalized discrete valuation that sends a uniformizer to 1.

We take $\zeta' \in k^\times$. We simply write $\pi_{\zeta'}$, $\Lambda_{\zeta'}$ and $\tau_{\zeta'}$ for $\pi_{\zeta', 1, 1}$, $\Lambda_{\zeta', 1, 1}$ and $\tau_{\zeta', 1, 1}$ respectively. We put $F = K(\zeta''\varpi^{1/3})$ and $L' = F(\varphi)$. We define $\mathfrak{J}_F, \mathfrak{P}_F \subset M_2(\mathcal{O}_F)$ similarly to \mathfrak{J} and \mathfrak{P} as in (4.1). We put $U_{\mathfrak{J}_F}^i = 1 + \mathfrak{P}_F^i$ for any positive integer i . We consider L' as a F -subalgebra of $M_2(F)$ similarly as (4.2). We put $\epsilon_{F/K} = (-1)^f$.

Let $\pi_{F, \zeta'}$ be the tame lifting of $\pi_{\zeta'}$ to F . See [BH, 46.5 Definition] for the tame lifting. We define a character $\Lambda_{F, \zeta'}: L'^\times U_{\mathfrak{J}_F}^2 \rightarrow \overline{\mathbb{Q}}_\ell^\times$ by $\Lambda_{F, \zeta'}(x) = \epsilon_{F/K}^{v_{L'}(x)} \Lambda_{\zeta'}(\text{Nr}_{L'/L}(x))$ for $x \in L'$ and $\Lambda_{F, \zeta'}(x) = (\psi_F \circ \text{tr})(\zeta'^{-2} \varphi^{-1}(x - 1))$ for $x \in U_{\mathfrak{J}_F}^2$. Then we have $\pi_{F, \zeta'} = \text{c-Ind}_{L' \times U_{\mathfrak{J}_F}^2}^{GL_2(F)} \Lambda_{F, \zeta'}$ by [BH, 46.3 Proposition] and the construction of the tame lifting.

We will describe the restriction of $\tau_{\zeta'}$ to W_F . The field F corresponds to the subgroup $Q_8 \rtimes \mathbb{Z}$ of $Q \rtimes \mathbb{Z}$.

First, we consider the case where f is even. We put $h_0(x) = x^2 - x$. Then we have $h_0(\delta^2 - \delta) \equiv 1/(\zeta''\varpi^{1/3}) \pmod{3/4}$. Hence we can take $\delta_2 \in F(\delta)$ such that $h_0(\delta_2) = 1/(\zeta''\varpi^{1/3})$ and $\delta_2 \equiv \delta^2 - \delta \pmod{3/4}$ by Newton's method. Similarly, we can take $\delta_4 \in F(\delta)$ such that $\delta_4^2 - \delta_4 = \delta_2$ and $\delta_4 \equiv \delta \pmod{3/4}$. Then we have $F(\delta_4) = F(\delta)$. Further, we can take $\theta_2 \in F(\theta)$ such that $\theta_2^2 - \theta_2 = \delta_4^3$ and $\theta_2 \equiv \theta \pmod{7/12}$. We have $F(\theta_2) = F(\theta)$. Then $F(\delta_2)$ corresponds to the subgroup $C_4 \rtimes \mathbb{Z}$ of $Q \rtimes \mathbb{Z}$.

Next, we consider the case where f is odd. We put $h_1(x) = x^2 - x + 1$. Then we have $h_1(\delta^2 - \delta + \zeta_3) \equiv 1/(\zeta''\varpi^{1/3}) \pmod{3/4}$. Hence we can take $\delta_2 \in F(\zeta_3, \delta)$ such that $h_1(\delta_2) = 1/(\zeta''\varpi^{1/3})$ and $\delta_2 \equiv \delta^2 - \delta + \zeta_3 \pmod{3/4}$ by Newton's method. Similarly, we can take $\delta_4 \in F(\zeta_3, \delta)$ such that $\delta_4^2 - \delta_4 + \zeta_3 = \delta_2$ and $\delta_4 \equiv \delta \pmod{3/4}$. Then we have $F(\zeta_3, \delta_4) = F(\zeta_3, \delta)$. Further, we can take $\theta_2 \in F(\zeta_3, \theta)$ such that $\theta_2^2 - \theta_2 = \delta_4^3$ and $\theta_2 \equiv \theta \pmod{7/12}$. We have $F(\zeta_3, \theta_2) = F(\zeta_3, \theta)$. Then $F(\delta_2)$ corresponds to the subgroup C of $Q \rtimes \mathbb{Z}$.

We put $E = F(\delta_2)$. Let $\phi_{\zeta'}$ be the character of W_E induced from ϕ_1 by $\Theta_{\zeta'}$. Then we have $\tau_{\zeta'}|_{W_F} \simeq \text{Ind}_{W_E}^{W_F} \phi_{\zeta'}$. We consider $\phi_{\zeta'}$ as a character of E^\times by the Artin reciprocity map Art_E . For a finite extension K' of K and integer i , we write $\mathfrak{p}_{K'}$ for the maximal ideal of $\mathcal{O}_{K'}$, and put $U_{K'}^i = 1 + \mathfrak{p}_{K'}^i$. Let E_m be the unramified extension over E of degree m for a positive integer m . Let $\varkappa_{E/F}$ be the character of F^\times with kernel $\text{Nr}_{E/F}(E^\times)$.

For a Galois group G of a finite Galois extension of a non-Archimedean field, let $G_{(s)}$ and $G^{(t)}$ be the ramification subgroups of G with lower numbering and upper numbering.

Lemma 5.2. *We have $\phi_{\zeta'}(1+x) = \psi_E(\delta_2^3 x)$ for $x \in \mathfrak{p}_E^2$ and $\varkappa_{E/F}(1+y) = \psi_F((\zeta''\varpi^{1/3})^{-1}y)$ for $y \in \mathfrak{p}_F$.*

Proof. We prove the first statement only in the case where f is odd. It is easier to prove the first statement in the case where f is even.

We put $G = \text{Gal}(E_2(\theta)/E)$. For $\sigma \in I_E$, we can show that

$$v\left(\sigma\left(\frac{\delta}{\theta}\right) - \frac{\delta}{\theta}\right) = \begin{cases} \frac{1}{12} & \text{if } \zeta_{3, \sigma} = 1, \nu_\sigma = 1, \\ \frac{1}{6} & \text{if } \zeta_{3, \sigma} = 1, \nu_\sigma = 0, \mu_\sigma = 1 \end{cases}$$

by the definition of ζ_σ , ν_σ and μ_σ . Then we have $\text{Gal}(E_2(\theta)/E_2) = G_{(0)} = G_{(1)} \supset \text{Gal}(E_2(\theta)/E_2(\delta)) = G_{(2)} = G_{(3)} \supset \{1\} = G_{(4)}$ and

$$G^{(t)} = \begin{cases} \text{Gal}(E_2(\theta)/E_2) & \text{if } 0 \leq t \leq 1, \\ \text{Gal}(E_2(\theta)/E_2(\delta)) & \text{if } 1 < t \leq 2, \\ \{1\} & \text{if } 2 < t. \end{cases}$$

Then the restriction of $\phi_{\zeta'}$ to U_E^2 is the homomorphism

$$U_E^2 \twoheadrightarrow U_E^2/(U_E^3 \text{Nr}_{E_2(\theta)/E}(U_{E_2(\theta)}^3)) \xrightarrow{\sim} \text{Gal}(E_2(\theta)/E_2(\delta)) \simeq Z \xrightarrow{\phi|_Z} \overline{\mathbb{Q}}_\ell^\times$$

by [Se, XV §2 Corollaire 3 au Théorème 1]. We define $N_2: k_2 \rightarrow k$ by $N_2(x) = \text{Tr}_{k_2/k}(x)^2 + \text{Tr}_{k_2/k}(x)$. Then we can check that $\text{Nr}_{E_2(\theta)/E}: U_{E_2(\theta)}^3/U_{E_2(\theta)}^4 \rightarrow U_E^2/U_E^3$ becomes $N_2: k_2 \rightarrow k$ under the identifications

$$U_{E_2(\theta)}^3/U_{E_2(\theta)}^4 \simeq k_2; 1 + \theta^{-1}x \mapsto \bar{x} \text{ and } U_E^2/U_E^3 \simeq k; 1 + \delta_2^{-2}x \mapsto \bar{x}.$$

Therefore we have $\phi_{\zeta'}(1+x) = (\chi_2 \circ \text{Tr}_{k/\mathbb{F}_2})(\overline{\delta_2^2 x})$, because $\text{Im } N_2 = \text{Ker } \text{Tr}_{k/\mathbb{F}_2}$. Since we have $(\chi_2 \circ \text{Tr}_{k/\mathbb{F}_2})(\bar{x}) = \psi_E(\delta_2 x)$, the first statement follows.

We can prove the second statement similarly. \square

Lemma 5.3. *We consider $\hat{\zeta}'^{-2}\varphi^{-1}$ as an element of $GL_2(F)$ by the embedding (4.2). Then we have*

$$\begin{aligned} \det(\hat{\zeta}'^{-2}\varphi^{-1}) &\equiv \text{Nr}_{E/F}(\delta_2^3) \pmod{U_F^1}, \\ \text{tr}(\hat{\zeta}'^{-2}\varphi^{-1}) &\equiv (\zeta''\varpi^{\frac{1}{3}})^{-1} + \text{Tr}_{E/F}(\delta_2^3) \pmod{\mathcal{O}_F}. \end{aligned}$$

Proof. We can check the claim by easy calculations. \square

Let \mathcal{E}' be the elliptic curve over \mathbb{F}_2 defined by $z^2 + z = w^3 + w$. Let $\eta_2, \eta'_2 \in \overline{\mathbb{Q}}_\ell$ be the roots of $x^2 + 2x + 2 = 0$.

Lemma 5.4. *We have $|\mathcal{E}(\mathbb{F}_q)| = q + 1 - ((-2)^{1/2})^f - (-(-2)^{1/2})^f$ and $|\mathcal{E}'(\mathbb{F}_q)| = q + 1 - \eta_2^f - \eta'_2{}^f$.*

Proof. We have proved $\text{tr}(\text{fr}_q^*; H^1(\mathcal{E}_{k^{\text{ac}}}, \overline{\mathbb{Q}}_\ell)) = ((-2)^{1/2})^f + (-(-2)^{1/2})^f$ in the first paragraph of the proof of Proposition 3.5. The first claim follows from this. The second claim is proved similarly. \square

If f is odd, then the map $\Theta_{\zeta'}$ induces an isomorphism $W(E^{\text{ur}}(\theta)/E) \simeq C$, and we write \mathbf{a}_E for the composite

$$E^\times \xrightarrow{\text{Art}_E} W_E^{\text{ab}} \twoheadrightarrow W(E^{\text{ur}}(\theta)/E) \simeq C.$$

Lemma 5.5. *We assume that f is odd. Let n_f and m_f be the integers such that $1 \leq n_f, m_f \leq 2$, $n_f \equiv (f+1)/2 \pmod{2}$ and $m_f \equiv (f^2+7)/8 \pmod{2}$. Then we have $\mathbf{a}_E(\delta_2) = (g(1, \bar{\zeta}_3^{2n_f}, \bar{\zeta}_3^{m_f}), -1)$.*

Proof. Let \overline{C} be the image of C in $Q_8 \rtimes (\mathbb{Z}/2\mathbb{Z})$. Then \overline{C} is a cyclic group of order 8. Let $\overline{\mathbf{a}}_E$ be the composite of \mathbf{a}_E with the natural projection $C \rightarrow \overline{C}$. It suffices to show that $\overline{\mathbf{a}}_E(\delta_2) = (g(1, \bar{\zeta}_3^{2n_f}, \bar{\zeta}_3^{m_f}), 1)$, because we know that the second component of $\mathbf{a}_E(\delta_2)$ is -1 . We note that the isomorphism $W(E^{\text{ur}}(\theta)/E) \simeq C$ induces $\text{Gal}(E_2(\theta_2)/E) \simeq \overline{C}$. By this isomorphism, we consider $(g(1, \bar{\zeta}_3^{2n_f}, \bar{\zeta}_3^{m_f}), 1)$ as an element of $\text{Gal}(E_2(\theta_2)/E)$.

We write $E_{(0)}$ for E in the mixed characteristic case, and $E_{(p)}$ for E in the equal characteristic case. We use similar notations for other fields and elements of the fields. Then we have the isomorphism

$$E_{(0)}^\times/U_{E_{(0)}}^3 \simeq E_{(p)}^\times/U_{E_{(p)}}^3; \hat{\xi}_0 + \hat{\xi}_1\delta_{2,(0)}^{-1} + \hat{\xi}_2\delta_{2,(0)}^{-2} \mapsto \xi_0 + \xi_1\delta_{2,(p)}^{-1} + \xi_2\delta_{2,(p)}^{-2}$$

where $\xi_0, \xi_1, \xi_2 \in k \subset E_{(p)}$ and $\hat{\xi}_0, \hat{\xi}_1, \hat{\xi}_2$ are their lifts in $\mu_{q-1}(E_{(0)}) \cup \{0\}$. This isomorphism induces an isomorphism

$$(\text{Gal}(E_{(0)}^{\text{sep}}/E_{(0)})/\text{Gal}(E_{(0)}^{\text{sep}}/E_{(0)})^{(3)})^{\text{ab}} \simeq (\text{Gal}(E_{(p)}^{\text{sep}}/E_{(p)})/\text{Gal}(E_{(p)}^{\text{sep}}/E_{(p)})^{(3)})^{\text{ab}}$$

by [De, (3.5.2)]. It further induces an isomorphism $\text{Gal}(E_{2,(0)}(\theta_{2,(0)})/E_{(0)}) \simeq \text{Gal}(E_{2,(p)}(\theta_{2,(p)})/E_{(p)})$. Then we have a commutative diagram

$$\begin{array}{ccccc} E_{(0)}^\times/U_{E_{(0)}}^3 & \xrightarrow{\text{Art}_{E_{(0)}}} & \text{Gal}(E_{2,(0)}(\theta_{2,(0)})/E_{(0)}) & \xrightarrow{\sim} & \overline{C} \\ \wr \downarrow & & \wr \downarrow & & \parallel \\ E_{(p)}^\times/U_{E_{(p)}}^3 & \xrightarrow{\text{Art}_{E_{(p)}}} & \text{Gal}(E_{2,(p)}(\theta_{2,(p)})/E_{(p)}) & \xrightarrow{\sim} & \overline{C} \end{array}$$

by [De, (3.6.1)] and the construction of the isomorphisms. Therefore, it suffices to show that $\overline{\alpha}_E(\delta_2) = (g(1, \zeta_3^{2n_f}, \zeta_3^{m_f}), 1)$ in the equal characteristic case.

We assume that the characteristic of E is p . We define the central division algebra D_g over E of degree 64 by $D_g \simeq \bigoplus_{i=0}^7 E_2(\theta_2)s^i$ where $s^8 = \delta_2$ and $sas^{-1} = (g(1, \zeta_3^{2n_f}, \zeta_3^{m_f}), 1)(a)$ for $a \in E_2(\theta_2)$. Let $\sigma_q \in \text{Gal}(E_8/E)$ be the lift of Fr_q . We define the central division algebra D_σ over E of degree 64 by $D_\sigma \simeq \bigoplus_{i=0}^7 E_8 t^i$ where $t^8 = \delta_2$ and $tat^{-1} = \sigma_q(a)$ for $a \in E_8$. By the construction of the Artin reciprocity map, it suffices to show $D_g \simeq D_\sigma$ to prove the claim. To show this isomorphism, it suffices to find $s', \delta'_4, \theta'_2 \in D_\sigma$ such that

$$\begin{aligned} s'^8 &= \delta_2, & \delta_4'^2 - \delta_4' + \zeta_3 &= \delta_2, & \theta_2'^2 - \theta_2' &= \delta_4'^3, & \delta_4' \theta_2' &= \theta_2' \delta_4', \\ s' \zeta_3 s'^{-1} &= \zeta_3^2, & s' \delta_4' s'^{-1} &= \delta_4' + \zeta_3^{n_f}, & s' \theta_2' s'^{-1} &= \theta_2' + \zeta_3^{2n_f} \delta_4' + \zeta_3^{m_f}. \end{aligned}$$

We put $s' = t$. Then $s'^8 = \delta_2$ and $s' \zeta_3 s'^{-1} = \zeta_3^2$. We take $a_0 \in \mu_{q^4-1}(E_4)$ such that $a_0^2 - a_0 = \zeta_3$. We put $\delta_4' = a_0 + t^2 + t^4$. Then we can check that $\delta_4'^2 - \delta_4' + \zeta_3 = \delta_2$ using $t^2 a_0 t^{-2} = a_0 + 1$. We can check also that $t \delta_4' t^{-1} = \delta_4' + \zeta_3^{n_f}$ using $t a_0 t^{-1} = a_0 + \zeta_3^{n_f}$.

We take $b_0 \in \mu_{q^8-1}(E_8)$ and $b_4 \in \mu_{q^4-1}(E_4)$ such that $b_0^2 - b_0 = a_0 \zeta_3^2 + \zeta_3$ and $b_4^2 = a_0$. We put $\theta_2' = b_0 + (a_0 + \zeta_3)t^2 + b_4 t^4 + t^6$. Then we can check that $\theta_2'^2 - \theta_2' = \delta_4'^3$, $\delta_4' \theta_2' = \theta_2' \delta_4'$ and $t \theta_2' t^{-1} = \theta_2' + \zeta_3^{2n_f} \delta_4' + \zeta_3^{m_f}$ using $t b_0 t^{-1} = b_0 + a_0 \zeta_3^{2n_f} + \zeta_3^{m_f}$ and $t b_4 t^{-1} = b_4 + \zeta_3^{2n_f}$. Therefore, we have proved the claim. \square

In the next proposition, we show that $\tau_{\zeta'}|_{W_F}$ corresponds to $\pi_{F,\zeta'}$ by the local Langlands correspondence by calculating epsilon factors. We will show a correspondence over K in Theorem 5.7 using this correspondence over F .

Proposition 5.6. *We have $\text{LL}_\ell(\tau_{\zeta'}|_{W_F}) = \pi_{F,\zeta'}$.*

Proof. We put $\text{LL}_\ell(\tau_{\zeta'}) = \pi'_{\zeta'}$ and $\text{LL}_\ell(\tau_{\zeta'}|_{W_F}) = \pi'_{F,\zeta'}$. We want to show that $\pi'_{F,\zeta'} = \pi_{F,\zeta'}$.

By Lemma 5.2, Lemma 5.3 and [BH, 44.7 Proposition], the representation $\pi'_{F,\zeta'}$ contains the ramified simple stratum $(\mathfrak{J}_F, 3, \hat{\zeta}'^{-2}\varphi^{-1})$. Then the representation $\pi'_{\zeta'}$ contains the ramified simple stratum $(\mathfrak{J}, 1, \hat{\zeta}'^{-2}\varphi^{-1})$ by the construction of $\pi'_{\zeta'}$ in [BH, 50.3]. Therefore we have $\pi'_{\zeta'} = \text{c-Ind}_{L^\times U_{\mathfrak{J}}^1}^{GL_2(K)} \Lambda'_{\zeta'}$ for a character $\Lambda'_{\zeta'}: L^\times U_{\mathfrak{J}}^1 \rightarrow \overline{\mathbb{Q}}_\ell^\times$ such that $\Lambda'_{\zeta'} = \Lambda_{\zeta'}$ on $U_{\mathfrak{J}}^1$.

Let 1_F denote the trivial character of W_F . We put $\varkappa_{F/K} = \det \text{Ind}_{W_F}^{W_K} 1_F$. Then $\varkappa_{F/K}|_{W_F} = 1_F$ if f is even, and $\varkappa_{F/K}|_{W_F}$ is the unramified character of order two if f is odd. Hence, the definition of $\epsilon_{F/K}$ in [BH, 46.3] coincides with that in this paper. By [BH, 46.3 Proposition], we have $\pi'_{F,\zeta'} = \text{c-Ind}_{L' \times U_{\mathfrak{J}_F}^2}^{GL_2(F)} \Lambda'_{F,\zeta'}$ for a character $\Lambda'_{F,\zeta'}: L' \times U_{\mathfrak{J}_F}^2 \rightarrow \overline{\mathbb{Q}}_\ell^\times$ such that $\Lambda'_{F,\zeta'} = \Lambda_{F,\zeta'}$ on $U_{\mathfrak{J}_F}^2$ and $\Lambda'_{F,\zeta'}(x) = \epsilon_{F/K}^{v_{L'}(x)} \Lambda'_{\zeta'}(\text{Nr}_{L'/L}(x))$ for $x \in L'$. Hence we have $\Lambda'_{F,\zeta'}(x) = 1$ for $x \in U_{L'}^1$, because $\Lambda'_{\zeta'}(x) = 1$ for $x \in U_L^1$. Then we see that $\Lambda'_{F,\zeta'} = \Lambda_{F,\zeta'}$ on $x \in U_L^1 F^\times U_{\mathfrak{J}_F}^2$, because $\Lambda'_{F,\zeta'} = \Lambda_{F,\zeta'}$ on F^\times by Remark 5.1 and Lemma 3.4.

We define $\kappa_F: F^\times \rightarrow \overline{\mathbb{Q}}_\ell^\times$ by $\kappa_F(x) = (-1)^{v_F(x)}$. Since $\Lambda'_{F,\zeta'} = \Lambda_{F,\zeta'}$ on $x \in U_L^1 F^\times U_{\mathfrak{J}_F}^2$, we know that $\Lambda'_{F,\zeta'} = \Lambda_{F,\zeta'}$ or $\Lambda'_{F,\zeta'} = \Lambda_{F,\zeta'} \otimes (\kappa_F \circ \det)$. We take an isomorphism $\iota: \overline{\mathbb{Q}}_\ell \simeq \mathbb{C}$. Then, to show $\Lambda'_{F,\zeta'} = \Lambda_{F,\zeta'}$, it suffices to show that $\varepsilon({}^t \pi'_{F,\zeta'}, 1/2, \iota \circ \psi_F) = \varepsilon({}^t \pi_{F,\zeta'}, 1/2, \iota \circ \psi_F)$. We note that we have already known this equality up to sign.

In the sequel of this proof, we identify $\overline{\mathbb{Q}}_\ell$ and \mathbb{C} by ι , and omit to write ι . By [BH, 25.5 Corollary], we obtain $\varepsilon(\pi_{F,\zeta'}, 1/2, \psi_F) = -\epsilon_{F/K}$ using that 5 is the least integer $m \geq 0$ such that $U_{\mathfrak{J}_F}^{m+1} \subset \text{Ker } \Lambda_{F,\zeta'}$. On the other hand, we have $\varepsilon(\pi'_{F,\zeta'}, 1/2, \psi_F) = \varepsilon(\tau_{\zeta'}|_{W_F}, 0, \psi_F) = q^{3/2} \varepsilon(\tau_{\zeta'}|_{W_F}, 1/2, \psi_F)$, because the Artin conductor of $\tau_{\zeta'}|_{W_F}$ is three. Hence, it suffices to show that $\varepsilon(\tau_{\zeta'}|_{W_F}, 1/2, \psi_F) = -\epsilon_{F/K} q^{-3/2}$.

Let $\lambda_{E/F}(\psi_F)$ be the local constant of E over F with respect to ψ_F . Let 1_E denote the trivial character of W_E . Then we have

$$\lambda_{E/F}(\psi_F) = \varepsilon(\text{Ind}_{W_E}^{W_F} 1_E, 1/2, \psi_F) \varepsilon(1_E, 1/2, \psi_E)^{-1} = \varepsilon(\kappa_{E/F}, 1/2, \psi_F) = \kappa_{E/F}(\zeta'' \varpi^{1/3}) = \epsilon_{F/K}$$

by Lemma 5.2 and [BH, 23.6 Proposition], because we have

$$\text{Nr}_{E/F}(\delta_2) = \begin{cases} -1/(\zeta'' \varpi^{1/3}) & \text{if } f \text{ is even,} \\ -1/(\zeta'' \varpi^{1/3}) + 1 & \text{if } f \text{ is odd.} \end{cases}$$

Therefore, it suffices to show that $\varepsilon(\phi_{\zeta'}, 1/2, \psi_E) = -q^{-3/2}$, because we have $\varepsilon(\tau_{\zeta'}|_{W_F}, 1/2, \psi_F) = \varepsilon(\phi_{\zeta'}, 1/2, \psi_E) \lambda_{E/F}(\psi_F)$.

We define ψ'_E by $\psi'_E(x) = \psi_E(\delta_2 x)$ for $x \in E^\times$. Then ψ'_E has level one, and we have

$$\begin{aligned} \varepsilon(\phi_{\zeta'}, 1/2, \psi_E) &= \phi_{\zeta'}(\delta_2)^{-1} \varepsilon(\phi_{\zeta'}, 1/2, \psi'_E) = q^{-1/2} \phi_{\zeta'}(\delta_2)^{-1} \sum_{y \in U_E^1/U_E^2} \phi_{\zeta'}(\delta_2^2 y)^{-1} \psi'_E(\delta_2^2 y) \\ &= q^{-1/2} \phi_{\zeta'}(\delta_2^3)^{-1} \sum_{x \in \mathfrak{p}_E/\mathfrak{p}_E^2} \phi_{\zeta'}(1+x)^{-1} \psi_E(\delta_2^3 x) \end{aligned}$$

by [BH, 23.5 Lemma 1, (23.6.2) and (23.6.4)] and $\psi_E(\delta_2^3) = 1$. Therefore it suffices to show that

$$\phi_{\zeta'}(\delta_2^3)^{-1} \sum_{x \in \mathcal{O}_E/\mathfrak{p}_E} \phi_{\zeta'}(1 + \delta_2^{-1} x)^{-1} \psi_E(\delta_2^2 x) = -q^{-1}. \quad (5.1)$$

Note that we have already known this equality up to sign.

First, we consider the case where f is even. Then we have $\phi_{\zeta'}(\delta_2^3) = (-2)^{3f/2}$ by $\text{Nr}_{E(\theta_2)/E}(\theta_2) = -\delta_2^3$ and Lemma 5.2. Hence, it suffices to show $\sum_{x \in \mathcal{O}_E/\mathfrak{p}_E} \phi_{\zeta'}(1 + \delta_2^{-1} x)^{-1} \psi_E(\delta_2^2 x) = -(-2)^{f/2}$. For $\xi \in k$, the Teichmüller lift of ξ is denoted by $\hat{\xi}$. We have

$$\phi_{\zeta'}(1 + (\hat{\xi}^2 + \hat{\xi}^4) \delta_2^{-1} + (\hat{\xi} + \hat{\xi}^2 + \hat{\xi}^3) \delta_2^{-2}) = 1, \quad \phi_{\zeta'}(1 + (\hat{\xi}^2 + \hat{\xi}^4) \delta_2^{-2}) = 1 \quad (5.2)$$

for $\xi \in k$, since

$$\begin{aligned} \text{Nr}_{E(\theta_2)/E}(1 + \hat{\xi} \delta_4 \theta_2^{-1}) &\equiv 1 + (\hat{\xi}^2 + \hat{\xi}^4) \delta_2^{-1} + (\hat{\xi} + \hat{\xi}^2 + \hat{\xi}^3) \delta_2^{-2} \pmod{1/2}, \\ \text{Nr}_{E(\theta_2)/E}(1 + \hat{\xi} \delta_4^2 \theta_2^{-2}) &\equiv 1 + (\hat{\xi}^2 + \hat{\xi}^4) \delta_2^{-2} \pmod{1/2}. \end{aligned}$$

Therefore we see that $\phi_{\zeta'}(1 + \delta_2^{-1} x) = \pm \sqrt{-1}$ for $x \in \mathcal{O}_E$ if $\bar{x} \neq \xi^2 + \xi^4$ for any $\xi \in k$. Then we have

$$\sum_{x \in \mathcal{O}_E/\mathfrak{p}_E} \phi_{\zeta'}(1 + \delta_2^{-1} x)^{-1} \psi_E(\delta_2^2 x) = \frac{1}{2} \sum_{\xi \in k} \phi_{\zeta'}(1 + (\hat{\xi}^2 + \hat{\xi}^4) \delta_2^{-1})^{-1} \psi_E(\delta_2^2 (\hat{\xi}^2 + \hat{\xi}^4)),$$

since we have already known that $\sum_{x \in \mathcal{O}_E/\mathfrak{p}_E} \phi_{\zeta'}(1 + \delta_2^{-1} x)^{-1} \psi_E(\delta_2^2 x) = \pm (-2)^{f/2} \in \mathbb{Q}$. We have

$$\sum_{\xi \in k} \phi_{\zeta'}(1 + (\hat{\xi}^2 + \hat{\xi}^4) \delta_2^{-1})^{-1} \psi_E(\delta_2^2 (\hat{\xi}^2 + \hat{\xi}^4)) = \sum_{\xi \in k} \phi_{\zeta'}(1 + \hat{\xi}^3 \delta_2^{-2}) \psi_E(\delta_2^2 (\hat{\xi}^2 + \hat{\xi}^4)),$$

since

$$\phi_{\zeta'}(1 + (\hat{\xi}^2 + \hat{\xi}^4) \delta_2^{-1})^{-1} = \phi_{\zeta'}(1 + (\hat{\xi} + \hat{\xi}^2 + \hat{\xi}^3) \delta_2^{-2}) = \phi_{\zeta'}(1 + \hat{\xi}^3 \delta_2^{-2})$$

by (5.2). Further we have

$$\begin{aligned} \sum_{\xi \in k} \phi_{\zeta'}(1 + \hat{\xi}^3 \delta_2^{-2}) \psi_E(\delta_2^2(\hat{\xi}^2 + \hat{\xi}^4)) &= \sum_{\xi \in k} \chi_2(\text{Tr}_{k/\mathbb{F}_2}(\xi^3 + \xi^2 + \xi^4)) \\ &= \sum_{\xi \in k} \chi_2(\text{Tr}_{k/\mathbb{F}_2}(\xi^3)) = -2(-2)^{f/2} \end{aligned}$$

by Lemma 5.2, because $\text{Ker Tr}_{k/\mathbb{F}_2} = \{\xi + \xi^2 \mid \xi \in k\}$ and

$$|\{(x, y) \in k^2 \mid x^2 + x = y^3\}| = |\mathcal{E}(\mathbb{F}_q)| - 1 = q - 2(-2)^{f/2}$$

by Lemma 5.4. Thus we have the claim in the case where f is even.

Next we consider the case where f is odd. We define n_f and m_f as in Lemma 5.5. We treat only the case where $n_f = m_f = 1$. The other cases are proved similarly.

We assume that $n_f = m_f = 1$, which is equivalent to that $f \equiv 1 \pmod{8}$. By Lemma 5.5 and the definition of $\phi_{\zeta'}$, we have $\phi_{\zeta'}(\delta_2) = q/\eta$. Hence, to prove (5.1), it suffices to show $\sum_{x \in \mathcal{O}_E/\mathfrak{p}_E} \phi_{\zeta'}(1 + \delta_2^{-1}x)^{-1} \psi_E(\delta_2^2 x) = -q^2 \eta^{-3}$. We have

$$\phi_{\zeta'}\left(1 + \text{Tr}_{E_2/E}(\hat{\xi}^2 + \hat{\xi}^4) \delta_2^{-1} + (\text{Tr}_{E_2/E}(\hat{\xi} + \hat{\xi}^2 \zeta_3^2 + \hat{\xi}^3 + \hat{\xi}^4 \zeta_3) + \text{Nr}_{E_2/E}(\hat{\xi}^2 + \hat{\xi}^4)) \delta_2^{-2}\right) = 1, \quad (5.3)$$

$$\phi_{\zeta'}(1 + \text{Tr}_{E_2/E}(\hat{\xi}^2 + \hat{\xi}^4) \delta_2^{-2}) = 1 \quad (5.4)$$

for $\xi \in k_2$, since

$$\begin{aligned} \text{Nr}_{E_2(\theta_2)/E}(1 + \hat{\xi} \delta_4 \theta_2^{-1}) &\equiv 1 + \text{Tr}_{E_2/E}(\hat{\xi}^2 + \hat{\xi}^4) \delta_2^{-1} \\ &\quad + (\text{Tr}_{E_2/E}(\hat{\xi} + \hat{\xi}^2 \zeta_3^2 + \hat{\xi}^3 + \hat{\xi}^4 \zeta_3) + \text{Nr}_{E_2/E}(\hat{\xi}^2 + \hat{\xi}^4)) \delta_2^{-2} \pmod{1/2}, \\ \text{Nr}_{E_2(\theta_2)/E}(1 + \hat{\xi} \delta_4^2 \theta_2^{-2}) &\equiv 1 + \text{Tr}_{E_2/E}(\hat{\xi}^2 + \hat{\xi}^4) \delta_2^{-2} \pmod{1/2}. \end{aligned}$$

By (5.4), we have

$$\phi_{\zeta'}(1 + (\hat{\xi} + \hat{\xi}^2) \delta_2^{-2}) = 1 \quad (5.5)$$

for $\xi \in k$, because $\text{Tr}_{k_2/k}$ is surjective and $\text{Tr}_{k_2/k}(\xi^2 + \xi^4) = \text{Tr}_{k_2/k}(\xi)^2 + \text{Tr}_{k_2/k}(\xi)^4$ for $\xi \in k_2$. Then, by (5.3) and (5.5), we have

$$\phi_{\zeta'}(1 + (\hat{\xi}^2 + \hat{\xi}^4) \delta_2^{-1} + (\hat{\xi} + \hat{\xi}^3) \delta_2^{-2}) = 1 \quad (5.6)$$

for $\xi \in k$, because

$$\begin{aligned} \text{Tr}_{k_2/k}(\xi^3) + \text{Nr}_{k_2/k}(\xi^2 + \xi^4) &= \text{Tr}_{k_2/k}(\xi)^3 + (\text{Nr}_{k_2/k}(\xi^2) + \xi^{q+1} \text{Tr}(\xi)) \\ &\quad + (\text{Nr}_{k_2/k}(\xi^2) + \xi^{q+1} \text{Tr}(\xi))^2 \end{aligned}$$

for $\xi \in k_2$. We have $\phi_{\zeta'}(\delta_2^2 + \delta_2 + 1) = -q$, since $\text{Nr}_{E_2(\theta_2)/E}(\theta_2/\delta_4) = \delta_2^2 + \delta_2 + 1$. Hence, we obtain $\phi_{\zeta'}(1 + \delta_2^{-1} + \delta_2^{-2}) = -\eta^2 q^{-1}$. Therefore we have

$$\phi_{\zeta'}(1 + (\hat{\xi}^2 + \hat{\xi}^4 + 1) \delta_2^{-1})^{-1} \psi_E(\delta_2^2(\hat{\xi}^2 + \hat{\xi}^4 + 1)) = -\frac{q}{\eta^2} \phi_{\zeta'}(1 + (\hat{\xi}^2 + \hat{\xi}^4) \delta_2^{-1})^{-1} \psi_E(\delta_2^2(\hat{\xi}^2 + \hat{\xi}^4)),$$

since we have $\phi_{\zeta'}(1 + \delta_2^{-2}) = \psi_E(\delta_2) = \psi_E(\delta_2^2)$ by Lemma 5.2. Then we have

$$\sum_{x \in \mathcal{O}_E/\mathfrak{p}_E} \phi_{\zeta'}(1 + \delta_2^{-1}x)^{-1} \psi_E(\delta_2^2 x) = \frac{1}{2} \left(1 - \frac{q}{\eta^2}\right) \sum_{\xi \in k} \phi_{\zeta'}(1 + (\hat{\xi}^2 + \hat{\xi}^4) \delta_2^{-1})^{-1} \psi_E(\delta_2^2(\hat{\xi}^2 + \hat{\xi}^4)),$$

because $\xi \in \text{Ker Tr}_{k/\mathbb{F}_2} = \{\xi'^2 + \xi'^4 \mid \xi' \in k\}$ if and only if $\xi + 1 \notin \text{Ker Tr}_{k/\mathbb{F}_2}$. Therefore, it suffices to show that

$$\sum_{\xi \in k} \phi_{\zeta'}(1 + (\hat{\xi}^2 + \hat{\xi}^4) \delta_2^{-1})^{-1} \psi_E(\delta_2^2(\hat{\xi}^2 + \hat{\xi}^4)) = -(-2)^{(f+1)/2}.$$

On the other hand, by (5.6) and Lemma 5.2, we have

$$\begin{aligned} \sum_{\xi \in k} \phi_{\zeta'}(1 + (\hat{\xi}^2 + \hat{\xi}^4)\delta_2^{-1})^{-1} \psi_E(\delta_2^2(\hat{\xi}^2 + \hat{\xi}^4)) &= \sum_{\xi \in k} \phi_{\zeta'}(1 + (\hat{\xi} + \hat{\xi}^3)\delta_2^{-2}) \psi_E(\delta_2^2(\hat{\xi}^2 + \hat{\xi}^4)) \\ &= \sum_{\xi \in k} \chi_2(\text{Tr}_{k/\mathbb{F}_2}(\xi + \xi^3)) = -(-2)^{(f+1)/2} \end{aligned}$$

because $\text{Ker Tr}_{k/\mathbb{F}_2} = \{\xi + \xi^2 \mid \xi \in k\}$ and

$$|\{(x, y) \in k^2 \mid x^2 + x = y^3 + y\}| = |\mathcal{E}'(\mathbb{F}_q)| - 1 = q - (-2)^{(f+1)/2}$$

by Lemma 5.4 under the assumption $f \equiv 1 \pmod{8}$. Thus we have proved the claim. \square

Theorem 5.7. *For $\zeta' \in k^\times$, $\chi \in (k^\times)^\vee$ and $c \in \overline{\mathbb{Q}}_\ell^\times$, we have $\text{LL}_\ell(\tau_{\zeta', \chi, c}) = \pi_{\zeta', \chi, c}$.*

Proof. We may assume that $\chi = 1$ and $c = 1$ by character twists. By [BH, 50.3 Lemma 1 and (50.3.2)] and Proposition 5.6, it suffices to show that $\Lambda_{\zeta'}|_{K^\times} \circ \text{Art}_K^{-1} = (\det \tau_{\zeta'}) \otimes |\cdot|^{-1}$, $\Lambda_{\zeta'}^3|_{U_3^1} = \Lambda_{F, \zeta'}|_{U_3^1}$ and $\Lambda_{\zeta'}^3(x) = \epsilon_{F/K}^{v_{L'}(x)} \Lambda_{F, \zeta'}(x)$ for $x \in L^\times$. The first equality follows from Lemma 3.4 and Remark 5.1, and the third equality follows from the definition of $\Lambda_{F, \zeta'}$.

We are going to show the second equality. We put

$$U_3' = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathfrak{J} \mid a \equiv d \equiv 1, b \equiv 0 \pmod{\mathfrak{p}} \right\}, \quad U_3'' = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in \mathfrak{J} \right\}.$$

Then we have $\Lambda_{\zeta'}^3|_{U_3'} = \Lambda_{F, \zeta'}|_{U_3'}$ by the definition of $\Lambda_{\zeta'}$ and $\Lambda_{F, \zeta'}$, because $U_3' \subset U_{3F}'$. On the other hand, we have

$$\begin{aligned} \Lambda_{F, \zeta'} \left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right) &= \Lambda_{F, \zeta'} \left(\varphi \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \varphi^{-1} \right) = \Lambda_{F, \zeta'} \left(\begin{pmatrix} 1 & 0 \\ b\varpi & 1 \end{pmatrix} \right) = \psi_F(\hat{\zeta}'^{-2}b) \\ &= \psi_K(\hat{\zeta}'^{-2}b)^3 = \Lambda_{\zeta'} \left(\begin{pmatrix} 1 & 0 \\ b\varpi & 1 \end{pmatrix} \right)^3 = \Lambda_{\zeta'} \left(\varphi \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \varphi^{-1} \right)^3 = \Lambda_{\zeta'} \left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right)^3 \end{aligned}$$

for $b \in \mathcal{O}_K$. Therefore we have the claim, because U_3^1 is generated by U_3' and U_3'' . \square

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